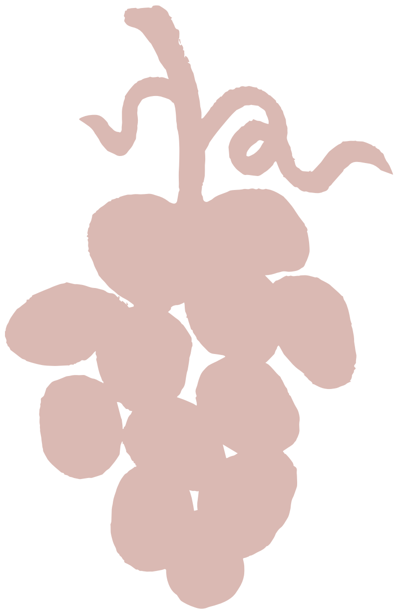
Doctoral Thesis



PARAMETRISING CLUSTERS OF SECTIONS

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No digáis que, agotado su tesoro,
de asuntos falta, enmudeció la lira;
podrá no haber poetas; pero siempre
habrá poesía.

Mientras las ondas de la luz al beso
palpiten encendidas,
mientras el sol las desgarradas nubes
de fuego y oro vista,
mientras el aire en su regazo lleve
perfumes y armonías,
mientras haya en el mundo primavera,
¡habrá poesía!

Mientras la ciencia a descubrir no alcance
las fuentes de la vida,
y en el mar o en el cielo haya un abismo
que al cálculo resista,
mientras la humanidad siempre avanzando
no sepa a dó camina,
mientras haya un misterio para el hombre,
¡habrá poesía!

Mientras se sienta que se ríe el alma,
sin que los labios rían;
mientras se llore, sin que el llanto acuda
a nublar la pupila;
mientras el corazón y la cabeza
batallando prosigan,
mientras haya esperanzas y recuerdos,
¡habrá poesía!

Mientras haya unos ojos que reflejen
los ojos que los miran,
mientras responda el labio suspirando
al labio que suspira,
mientras sentirse puedan en un beso
dos almas confundidas,
mientras exista una mujer hermosa,
¡habrá poesía!

–Gustavo Adolfo Becquer
1836 – 1870

To all my friends around the world and to my soulmate, *Àngels*

ABSTRACT

In this work we generalise clusters of points of a scheme to the relative setting, that is, we introduce *clusters of sections* of a family. When the family is smooth, we are able to show that there is a scheme parametrising its clusters of sections of length r. We called it the *universal scheme of clusters of sections* Cl^r. Such schemes are a generalisation of Kleiman's iterated blow ups (which parametrise clusters of points).

We present the first steps towards an iterative construction of the scheme Cl^{r+1} form Cl^r. We show that there is a morphism $F\colon \mathrm{Cl}^{r+1} \longrightarrow \mathrm{Cl}^r \times_{\mathrm{Cl}^{r-1}} \mathrm{Cl}^r$ (related to blowing up the diagonal) and a stratification of $\mathrm{Cl}_r \times_{\mathrm{Cl}_{r-1}} \mathrm{Cl}_r$ such that, via F, every irreducible component of Cl_{r+1} is either (a) birational to the closure of a stratum or (b) composed entirely of clusters whose $(r+1)$-th section is infinitely near to the r-th. Moreover, each type (a) irreducible component is a blow up of the closure of a stratum along a suitable sheaf of ideals, which fails to be Cartier only on the diagonal.

In order to clarify such iterative construction, we characterise the morphism F restricted to the union of type (a) irreducible components via a universal property. It is a generalisation of blow ups, which we call the *blow up split section family*. Roughly speaking, it combines the universal properties of blow ups and universal section families. We show that it exists under finite and projective conditions and that it exhibits some sort of birationality similar to F.

Meanwhile, we need to develop some auxiliary results and constructions lacking in the bibliography. For example, we show that, under certain assumptions, the blow up of a product of schemes along a locally principal subscheme preserve the product form. Given a family of schemes $\pi\colon X \longrightarrow Y$ and a morphism $\alpha\colon X \longrightarrow T$, we define (via a universal property) and prove the existence of a scheme parametrising those sections of π contained in some fibre of α. We also define (via a universal property) and prove the existence of a closed subscheme Z of Y such that α restricted to $X \times_Y Z$ is constant along the fibres of the projection $X \times_Y Z \longrightarrow Z$.

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Projecte que m'ha servit de bastió tots aquest anys.
Moltes gràcies per tot el temps que m'has dedicat.
Moltes gràcies per seguir allà quan jo no hi era.
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I per ultim i per això més, gràcies a Tu, la persona que ha fet i està fent la meva vida possible i preciosa.

CONTENTS

CONVENTIONS AND TERMINOLOGY

> "When I use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean –neither more nor less."
> "The question is," said Alice, "whether you can make words mean so many different things."
> "The question is," said Humpty Dumpty, "which is to be master –that's all."
>
> –Lewis Carroll
> *Through the Looking–Glass*

Let $\boldsymbol{C}$ be a small category and A, B objects of $\boldsymbol{C}$. We denote the set of arrows form A to B by $\boldsymbol{C}(A, B)$, the identity of A by $\mathbf{1}_A$ and the inverse of an isomorphism $f\colon A \longrightarrow B$ by f^{-1}. By a one-to-one correspondence between two sets we mean a one-to-one and onto map between them.

Let X, Y be schemes. We denote the underlying reduced subscheme of X by X_{red}. Given a point x of X, we denote by $\kappa(x)$ its residue field, by $\{x\}$ the scheme $\mathrm{Spec}(\kappa(x))$ and by $\overline{x}$ the schematic image of the natural morphism $\mathrm{Spec}(\kappa(x)) \hookrightarrow X$ (see Definition 1.46).

Given a morphism $f\colon X \longrightarrow Y$ and an open or closed subscheme Z of Y, we denote by $\mathrm{Im}(f)$ the schematic image of f, by $\mathrm{bl}(Z, Y)$ the blow up of Y along Z and by $f^{-1}(Z)$ the pullback of $Z \hookrightarrow Y$ by f (which is an open or closed subscheme of X). Given a section $\sigma\colon Y \longrightarrow X$ of f, we denote again by σ the closed subscheme of X image of σ. We denote by $X_\sigma \longrightarrow X$ the blow up of X along σ.

INTRODUCTION

> Infinitely near points are a nice and old idea for describing singularities.
>
> –Eduardo Casas-Alvero
> *Singularities of Plane Curves*

CLUSTERS OF SECTIONS

Infinitely near points appear already in the work of M. Nöther (introduced in [50, 51]) and their geometry was extensively developed by Enriques ([16, Book IV]). A modern account was given by Casas-Alvero in [7], introducing clusters of infinitely near points as the adequate notion to consider collections of infinitely near points. As usual in Algebraic Geometry, it becomes natural, and for some applications necessary, to study algebraic *families* of clusters of infinitely near points, which leads to the question of existence of universal parameter spaces for them; these are Kleiman's *iterated blowups* first introduced in [38]. This memoir deals with families of clusters and their parameter spaces in a relative setting, where *sections* of a family replace *points* of a variety or scheme. Doing so, we follow the general philosophy put forward by Grothendieck in his Éléments de Géometrie Algébrique", but at the same time we are motivated by possible applications to the study of linear systems, see Future work.

Given a separated morphism $\pi\colon X \longrightarrow Y$, an ordered cluster, or for simplicity a cluster, over π is an r-tuple $(t_1, \dots, t_r)$ where t_1 is a Y-point of $X_1 = X$, and where t_i, for $i \geqslant 1$, is a Y-point of the blow up[1] X_i of X_{i-1} along t_{i-1}. When t_i is, in fact, a Y-point of the exceptional divisor of X_i or of the pullback of the exceptional divisor of X_j for some $j \geqslant i$, we say that t_i is infinitely near to t_j. Since our definition differs from the standard one given in [7], some remarks on the differences may be in order.

- In [7], Y is the spectrum of the field of complex numbers, and X is a germ of complex surface. Generalising the definition to our relative context is straightforward, but it will become clear in the course of this work that some hypotheses on the morphism π are needed to obtain a well-behaved notion.

- When blowups commute, as is the case for blowups of surfaces along non infinitely near points, it is natural to identify clusters which differ only on the ordering of their points. Thus in the definition of [7] the

1 Classically, infinitely near points were defined by means of different birational transformations (e.g., the most used was ordinary quadratic transformations).

points forming the cluster form a set (with a partial order given by infinitely-near-ness) rather than an ordered tuple. However, for our purpose of later dealing with families of clusters, reorderings which may be admissible on general fibres need not extend to the whole family. For this reason it becomes more natural to work with the ordered version of clusters (and we follow the standard practice in the literature on families of clusters in doing so).

- Furthermore, [7] requires every point in a cluster to be infinitely near to the first one (which is in fact the reason for the choice of the word "cluster"). Again, since infinitely-near-ness between points may vary among the clusters in a family, imposing this condition in the definition of cluster becomes a burden when working in family and is usually avoided. The notion obtained by dropping it as we do is sometimes called "multi-cluster" but for simplicity we will adhere to the convention of [58] and call these objects simply clusters.

Let us now explain the notion of a family of clusters over π parametrised by a Y-scheme T, which we also call a T*-family of clusters over* π. It can be defined simply as a cluster $(t_1, \dots, t_r)$ over the base change of π to $X_T \longrightarrow T$. That is, a family of clusters is given by a sequence of blow ups

$$\begin{tikzcd}
(X_T)_{r+1} \arrow[r] & (X_T)_r \arrow[r] & \dots \arrow[r] & X_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T \arrow[d] \\
 & & & X \arrow[r] & Y,
\end{tikzcd}$$

where the centre $C_i \subseteq (X_T)_i$ of the blow up $(X_T)_{i+1} \longrightarrow (X_T)_i$ is the image of the section t_i of $(X_T)_i \longrightarrow T$. Given a Y-point of T, $Y \longrightarrow T$, and assuming when needed that blow ups commute with base changes[1], the Cartesian diagram

$$\begin{tikzcd}
X_{r+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \dots \arrow[r] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\
(X_T)_{r+1} \arrow[r] & (X_T)_r \arrow[r] & \dots \arrow[r] & X_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T \arrow[d] \\
 & & & X \arrow[r] & Y
\end{tikzcd}$$

illustrates how the pull back by $Y \longrightarrow T$ of the sequence of T-points $(t_1, \dots, t_r)$ is a cluster over π. Thus, Y-points of T *parametrise* clusters over π, and the set of all such parametrised clusters form the T-family.

Starting from the morphism π, Kleiman [38, Section 4.1, p.36] constructed inductively a sequence of (separated) morphisms $f_r : X_{r+1} \longrightarrow X_r$ for $r \geqslant 0$ as follows.[2] Define $f_0 : X_1 \longrightarrow X_0$ to be $\pi : X \longrightarrow Y$. Now, assume f_{r-1} defined. Consider the Cartesian product of X_r with itself over X_{r-1} and consider its

1 This is one of the main technicalities we face in this work.
2 We reproduce the construction word by word as it is done in [38].

diagonal subscheme Δ, which is a closed subscheme because f_{r-1} is separated. Define X_{r+1} to be the corresponding residual scheme and define f_r to be the composition of the structure map p and the second projection p_2.

$$
\begin{tikzcd}
 & X_{r+1} = R(\Delta, X_r \times_{X_{r-1}} X_r) \arrow[d, "p"] & \\
X_r \arrow[d, "f_{r-1}"'] & X_r \times_{X_{r-1}} X_r \arrow[l, "p_1"'] \arrow[d, "p_2"] \arrow[dl, phantom, "\lrcorner"] & \Delta \arrow[l, hook'] \\
X_{r-1} & X_r \arrow[l, "f_{r-1}"] &
\end{tikzcd}
$$

This construction[1] is commonly known as *Kleiman's iterated blow ups*. Kleiman's motivation came from some enumerative formulas for multiple points of the morphism π; later in [30, Proposition I.2, p.104], Harbourne realised that, in a different context[2], Kleiman's iterated blowups could be used as parameter spaces for rational surfaces with fixed Picard number, essentially by parametrising clusters of fixed length over the morphism $\pi : \mathbb{P}^2_{\Bbbk} \longrightarrow \operatorname{Spec}(\Bbbk)$.

The idea of using iterated blowups to parametrise clusters on smooth surfaces was further developed by Roé [58] for smooth surfaces $\pi : X \longrightarrow \operatorname{Spec}(\Bbbk)$, and by Kleiman and Piene [41] for smooth families $\pi : X \longrightarrow S$ of geometrically irreducible surfaces. It has found successful applications, especially in relative dimension 2, not only for its original motivation to enumerative geometry [38–40, 53, 54], but also in the study of linear systems of singular curves, [31, 56, 57, 59], of adjacencies of singularity types [1], or moduli spaces of polynomials [19]. The relations between Kleiman's iterated blow ups (the schemes X_r) and Hilbert schemes of points (the components of the Hilbert scheme $\mathcal{H}ilb_{X/Y}$ parametrising Y-points of X) became a recurring theme [17, 52, 56] which was clarified in [41].

Let now S be a ground scheme and assume that π is an S-morphism; in this situation, rather than a sequence of arbitrarily near Y-points of the Y-scheme X, a cluster over π can be understood as a sequence of arbitrarily near *sections* of the S-morphism π. With this perspective, a family of clusters over π parametrised by a S-scheme T must be a cluster of the base change $\pi_T : X_T \longrightarrow Y_T$ of π by $T \longrightarrow S$: this is what we call a T*-family of section-clusters over* π. So now, the objects parametrising the clusters in the family are the S-points of T, again if, when required, blow ups commute with base changes.

The whole purpose of this memoir is to develop the analogous machinery of Kleiman's iterated blowups in this relative setting. Our approach, taking the point of view of universal families and representable functors, is closest in spirit to that of [30] and [58], even though Harbourne goes even further

1 In [38], the schemes X_r are called r*-derived scheme of* $X \longrightarrow Y$.

2 An r-fold point of π is a point which has the same image as $r-1$ others. If π is an immersion, for instance, one expects finitely many such points, so that they can be counted; but then X has no Y-points and there are no clusters over π. In Harbourne's context instead, π is a submersion (smooth).

in the case of $\mathbb{P}^2$, considering isomorphisms between fibres of the universal family and the corresponding moduli problem, which gives rise to a quotient stack. We do not deal with these issues here.

Part of our representability results, with somewhat more restrictive hypotheses, have already appeared in print as [6]. An important intermediate step to explicitly build the universal families of section-clusters, which we call *the blow up split section family* (see below), has potential applications to other algebro-geometric problems. Its definition and existence are explained in the preprint [5], submitted for publication.

The organisation of this work in chapters is as follows. The main results are presented in the last chapter. The preceding chapters develop the techniques and constructions motivated by the problem of representing the functors under consideration, including the blow up split section family. We next describe in detail the content of each chapter; for the sake of motivation, we do it in reverse order.

Chapter 5. Clusters in family

Let S be a ground scheme and let $\pi\colon X \longrightarrow Y$ be an S-morphism. In Section 5.1, we give the formal definition of families of clusters of sections of π. We overcome the technicality that blow ups do not commute with arbitrary base changes by imposing regularity conditions on π, which lead us to the notion of *steady* family, see Definition 5.15. Under such conditions, families of section-clusters over π form a contravariant functor $\mathcal{Cl}^r\colon \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$, the functor of the parameter space problem for families of section-clusters, introduced in Section 5.2. When it is representable, we denote the representing scheme by Cl^r and we call it the r*-th universal scheme of section-clusters over* π (or r-Ucs for short). We present the functor $\mathcal{Cl}^r$ as a subfunctor of the Hilbert functor $\mathcal{Hilb}_{X/S}$ and we show that it is a subfunctor representable by locally closed embeddings, see Lemma 5.21. This way we reduce the representability of $\mathcal{Cl}^r$ to that of $\mathcal{Hilb}_{X/S}$, which gives the following existence theorem.

Theorem 5.19. *Let S be a ground scheme and $r \geqslant 1$ an integer. Let $\pi\colon X \longrightarrow Y$ be a steady S-family. If Y is proper and X is an at most countable disjoint union of quasiprojective schemes, then the r-Ucs Cl^r over π exists and the scheme Cl^r is an at most countable disjoint union of quasiprojective schemes.*

Kleiman's iterated blow ups X_r can be identified as our Cl^r when $Y \longrightarrow S$ is the identity, that is $Y = S$. In Section 5.4 we show that, when $Y \longrightarrow S$ is smooth, similarly to Kleiman's iterated blow ups, a recursive construction of Cl^{r+1} from Cl^r is possible.

More precisely, there is a morphism $F\colon \mathrm{Cl}^{r+1} \longrightarrow \mathrm{Cl}^r \times_{\mathrm{Cl}^{r-1}} \mathrm{Cl}^r$ and a stratification of $\mathrm{Cl}^r \times_{\mathrm{Cl}^{r-1}} \mathrm{Cl}^r$, where the diagonal Δ is a distinguished stratum, such that

Corollary 5.38.1. *Each irreducible component Z of Cl^{r+1} is either*

(a) *composed entirely of clusters whose* $(r+1)$*-th section is infinitely near to the* r*-th and* $F(Z) \subseteq \Delta$*,*

(b) *birational to an irreducible component of the closure* C *of a stratum, that is,* $F|_Z : Z \longrightarrow C$ *decomposes as* $Z \overset{i}{\hookrightarrow} \tilde{C} \longrightarrow C$ *where* i *is an open embedding and* $\tilde{C}$ *is a blowup of* C *whose centre fails to be Cartier only on* Δ*. In particular, if* $C \cap \Delta$ *is empty,* Z *is an open subscheme of* C*.*

We have a quite accurate explicit description of the strata, but not on the sheaf of ideals centre of such a blow up.

Chapter 4. The blow up split section family

Theorem 5.19 with more restrictive assumptions and Corollary 5.38.1 were firstly obtained by hand and published as [6]. In this work, they are more systematically presented, relying on a new notion, the *blow up split section family* (or blow up §family for short), which we introduce in Chapter 4 with the aim to shed light on such a recursive construction. In short, the morphism from the union of all type (a) irreducible components to the whole scheme $\mathrm{Cl}^r \times_{\mathrm{Cl}^{r-1}} \mathrm{Cl}^r$ is a blow up §family, which incorporates the stratification of $\mathrm{Cl}^r \times_{\mathrm{Cl}^{r-1}} \mathrm{Cl}^r$ and strata-wise it is the corresponding blow up (see Theorem 5.37 and Corollary 5.38.1).

We define the blow up §family and prove its existence in greater generality. Let X and Y be S-schemes and Z a closed subscheme of $X_Y = X \times_S Y$. The blow up §family of the projection $\pi : X_Y \longrightarrow Y$ along Z is a X-scheme $\mathfrak{B} \xrightarrow{b} X$ such that the pullback of Z by $(b \times \mathbf{1}_Y) : \mathfrak{B}_Y \longrightarrow X_Y$ is an effective Cartier divisor of $\mathfrak{B}_Y$ and satisfying a suitable universal property. Roughly speaking, it combines the universal properties of the universal section family, or Weil restriction, of π and of the blow up of X_Y along Z. When $Y = S$, we recover the classic blow up, but in general new phenomena may appear. For example, the resulting morphism $b \times \mathbf{1}_Y$ is not necessarily birational or even generically finite, see Section 4.1.

We prove that the blow up §family exists under some finiteness assumptions.

Theorem 4.3. *Assume all the schemes locally Noetherian. If* X_Y *is at most a countable disjoint union of quasiprojective schemes over* S*,* $X \longrightarrow S$ *is separated and* $Y \longrightarrow S$ *is proper, flat and with geometrically integral fibres, then the blow up §family of* π *along* Z *exists.*

Our proof of Corollary 5.38.1 stated above uses the following analogous result for blow up §families, which is our result on the structure for such morphisms. It generalises the fact that a blow up is an isomorphism away from its centre.

To state it we need to introduce some notation. Assume Y quasiprojective over S. Let $X = \sqcup_{\Phi \in \mathbb{Q}[t]} X_\Phi$ be the flattening stratification of the morphism $Z \longrightarrow X$. One of the strata, X_0 which we call the *core*, plays a special role, see Definition 4.7. Moreover, for every Φ, the points $x \in X_\Phi$ for which

$Z_x \hookrightarrow (X_Y)_x = Y_x$ is an effective Cartier divisor form a (possibly empty) open subscheme of X_Φ; we denote it by U_Φ.

Theorem 4.8. *Assume X connected, Y integral, Noetherian and projective and flat over S. Assume that the blow up §family $(\mathfrak{B}, b)$ of π along Z exists. Then, the open subscheme $\mathfrak{B} \setminus b^{-1}(X_0)$ of $\mathfrak{B}$ is isomorphic to $\sqcup_\Phi U_\Phi$.*

The chapter ends with some examples.

These results are contained in the preprint [5], submitted for publication.

Chapter 3. Building blocks

In Chapter 3, we introduce the f-constantify (or f-constfy for short) closed subscheme of Y –which is based on the functor *Iso* (see Definition 3.14) and the notion of $\aleph_1$-morphism (see Definitions 3.7 and 3.9)– and the universal split section family, fundamental steps for the construction of the blow up §family construction and the schemes Cl^r.

Let X, Y and W be S-schemes and let f be an S-morphism from $X_Y = X \times_S Y$ to W. The f-constfy closed subscheme Y' of Y satisfies that the restriction $f|_{X \times_Y Y'}$ is constant along the fibres of the projection $X \times_Y Y' \longrightarrow Y'$ plus a universal property. We prove its existence under weak assumptions.

Theorem 3.21. *Let S be a ground scheme. Let $p : X \longrightarrow Y$ and $f : X \longrightarrow W$ be S-morphisms. Consider the following Cartesian diagram.*

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[d, hook, "\Delta_{W/S}"] \\
X \times_Y X \arrow[r, "f \times_Y f"] & W \times_S W
\end{tikzcd}$$

Set $g : X \times_Y X \longrightarrow Y$. If W is separated over S and p is flat and proper, then the f-constfy closed subscheme of Y exists and it is the scheme representing the functor Iso_g^Z.

The existence of the f-constfy closed subscheme of Y follows from the representability of the functor *Iso*, which encodes the morphisms $T \longrightarrow Y$ such that $Z_T \hookrightarrow X \times_Y T$ is an isomorphism. The representability of this functor has been studied in the literature, but explicit constructions for the representing scheme are lacking. The class of $\aleph_1$-morphisms is introduced to fill this gap. The main property that allows an explicit description for the representing scheme of *Iso* is that arbitrary schematic unions commute with pullbacks by $\aleph_1$-morphisms, see Theorem 3.13.

Theorem 3.17. *Let $p : X \longrightarrow Y$ be a morphism and Z a closed subscheme of X. Let Ω denote the set of closed subschemes W of Y such that $Z_W \hookrightarrow X_W$ is an isomorphism and denote by Σ_Ω the closed subscheme $\Sigma_{W \in \Omega} W$ of Y. If p is $\aleph_1$-projective, then the scheme Σ_Ω represents the functor Iso_p^Z.*

The f-constfy construction allow us also to describe, for now set theoretically, where type (b) irreducible components of Cl^r emerge from, the ones missing in the blow up §family.

Coming back to the representability of the functor $\mathcal{Cl}^{\mathsf{T}}$, it is based on its identification with another functor we introduce, the *split section family* (see Definition 3.24). It is a subfunctor of the functor of sections representable by closed embeddings, which in turn is a subfunctor of a Hilbert functor representable by open embeddings. The goal of this construction is to parametrise sections of a morphism $\pi\colon X \longrightarrow Y$, but just those sections whose image is contained in some fibre of a morphism $\alpha\colon X \longrightarrow T$. We consider $\alpha\colon X \longrightarrow T$ as a morphism splitting the ambient space X by means of its fibres. So, the universal split section family is the scheme solving the parameter space problem of sections of π whose image is contained in some fibre of α.

Theorem 3.26. *If S is Noetherian, T is separated, X is at most a countable disjoint union of quasiprojective schemes over S, Y flat and proper over S, then the universal split section family of π exists and its underlying scheme is locally Noetherian and at most a countable disjoint union of quasiprojective schemes.*

Chapter 2. Technicalities on blow ups

The construction of the blow up §family consists of three steps, first we consider a blow up, second the universal split section family of such a blow up and, as a final step, we blow up a product of schemes $X \times_S Y$ along a locally principal subscheme, which we need to preserve the product form. Hence, we need to study when the product form is preserved under blow ups along locally principal subschemes, which we do in Section 2.1.

We formalise the idea that blowing up a locally Noetherian scheme along a locally principal subscheme consists in shaving off those associated points of the ambient scheme lying on the locally principal subscheme. We also show that, assuming $Y \longrightarrow S$ flat and with geometrically integral fibres, there is a one-to-one correspondence between the associated points of X and those of its base change $X \times_S Y$. This all yields the following result.

Theorem 2.8. *Assume all schemes are locally Noetherian. Let S be a ground scheme. Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be S-schemes. Let Z be a locally principal subscheme of $X \times_S Y$. Assume that $Y \xrightarrow{g} S$ is flat and with geometrically integral fibres. Then, there is a closed subscheme $i\colon W \longrightarrow X$ such that the closed embedding $i_Y\colon W \times_S Y \longrightarrow X \times_S Y$ is the blow up of $X \times_S Y$ along Z.*

If furthermore $Y \longrightarrow S$ is an fpqc morphism, for every S-morphism $T \xrightarrow{h'} X$ for which the preimage of Z by $h'_Y\colon T \times_S Y \longrightarrow X \times_S Y$ is an effective Cartier divisor, there is a unique morphism $h\colon T \longrightarrow W$ such that $i \circ h' = h$. Moreover, $h_Y\colon T \times_S Y \longrightarrow W \times_S Y$ is the morphism given by the universal property of the blow up i_Y.

Chapter 2 also includes an exposition of the conditions under which blow ups do commute with arbitrary base changes.

Chapter 1. Assorted preliminaries

Chapter 1 introduces the basic constructions, and the notation, widely used in later sections. It also includes an original result, Section 1.3.1, which explores the scheme-theoretic consequences of the set-theoretic definition of a constant morphism. Namely, a map $f\colon X \longrightarrow Y$ is constant if there is a unique $y \in Y$ such that $f(x) = y$ for every $x \in X$, in this case, if f is a morphism of schemes, we call f *naively constant through the point* y. We prove the following result.

Theorem 1.53. *Let $f\colon X \longrightarrow Y$ be a morphism with X quasi-compact. The morphism f is naively constant through a point y_0 of Y if and only if it factors through a morphism $Z \longrightarrow Y$ where the underlying topological space of Z is a point and the underlying reduced subscheme $\overline{Z}_{\mathrm{red}}$ of the schematic closure $\overline{Z}$ of $Z \longrightarrow Y$ is equal to the schematic closure of $\operatorname{Spec}(\kappa(y_0)) \hookrightarrow Y$. Moreover, if y_0 is a closed point of Y, then the morphism $Z \longrightarrow Y$ is a closed embedding.*

FUTURE WORK

> If we could first know where we are, and whither we are tending, we could then better judge what to do, and how to do it.
>
> –Abraham Lincoln
> *A House Divided*

Due to the technical complexity of some of our results, it hasn't been possible to attack within the time span of elaborating this thesis the problems which first motivated our study. We hope to be able in the near future to address at least some of them.

Theorem 5.37 asserts that the underlying topological space of Cl^{r+1} can be recovered from that of the blow up §family of $Cl^r \times_{Cl^{r-1}} Cl^r$ (along a suitable closed subscheme) and of the universal split section family of the exceptional divisor of X^r_{r+1} (see Definition 5.16). At the end of Section 5.4, we give evidences that the scheme structure of Cl^{r+1} may be recovered from such two schemes. We hope that studying the effect of blow ups of a projective scheme on its Hilbert scheme, which would be interesting on its own, could bring some insight on the problem.

The relationship between Kleiman's iterated blow ups and Hilbert schemes of points has been a recurring theme in applications, eventually clarified in [41]. It is natural to hope that the analogous forgetful maps (eliminating the ordering of the Y-points) from the universal schemes Cl_r will be useful in the study of the components of the Hilbert scheme $\mathcal{H}ilb_{X/S}$ parametrising sections of π. Some examples, like Example 5.42 and related computations, suggest that the schemes Cl^r can be especially useful in the study of Cohen-Macaulay Hilbert schemes.

As explained in [7], the whole set of infinitely near points of a Y-point t_1 of π provides a sort of infinitesimal space which displays the local geometry at t_1 of π. So, Kleiman's iterated blow ups encode all the infinitesimal information of the Y-scheme X, up to some order r, similarly to Semple towers (see [11, 12, 44]) or Jet schemes (see [13, 36] and Nash's original work [49]). It would be interesting to explore the connections between these different spaces, both in the absolute and in the relative settings.

As mentioned above, one of the successful applications of Kleiman's iterated blowups has been to the study of linear systems of singular curves; these applications often rely on the principle of semicontinuity, applied to an appropriate relative divisor on $X_{r+1} \longrightarrow X_r$ (see [31, 56, 57, 59]). Another successful approach in that context, introduced by Ciliberto and Miranda in [8, 9], is to apply the principle of semicontinuity to a relative divisor on a *degeneration* of the surface X, i.e., a family $X' \longrightarrow S$ whose general fibres are isomorphic to X and with a fibre which splits as a union of two or more components. It is then often simpler to analyse the system on the special fibre, as it splits in two or more “smaller” systems. Using a degeneration for such purposes involves the judicious choice of a set of sections of the degeneration; since our work provides a natural universal parameter space for such sets of sections (allowing infinitely near ones), it should be possible to combine both approaches to obtain a better understanding of linear systems of singular surfaces. The method of degenerations has also been applied to the computation of “collisions” (that is, adjacencies in the Hilbert scheme of points) in [10] but with important restrictions to the presence of infinitely near points in the general fibres. The techniques now available might also help eliminate such restrictions.

1 ASSORTED PRELIMINARIES

> The opening is where you plan your strategy. Where you place your initial stones determines the type of game you will play.
>
> –Richard Bozulich
> *The second book of GO*

This chapter introduces well-known constructions and sets notation, widely used in the later chapters.

1.1 CATEGORY THEORY.

This section introduces many constructions in Category Theory. Most of them will be used with no reference. For a more detailed treatment of Sections 1.1.1 and 1.1.2, we refer to [55] and [43] for modern and friendly references and to [46] as the classic reference. For a more detailed treatment of Section 1.1.3, we refer to [18, Chapter 2, pp.13–40].

Unless otherwise stated, by a category we mean a locally small category, that is, the collection of arrows between two objects always forms a set.

1.1.1 *Representability and universal properties*

> A related classical antecedent [to Yoneda's lemma] is a result that comforted those who were troubled by the abstract definition of a group: namely that any group is isomorphic to a subgroup of a permutation group
>
> –Emily Riehl
> *Category Theory in Context*

Representability of functors and universal properties (terminal and initial objects) are central for us. This section reviews these two notions, from Yoneda's lemmas to their equivalence via the category of elements and the constant functor.

Definition 1.1. A functor $\mathcal{F}: \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is *full* (resp. *faithful*) if for every pair of objects C and D of $\boldsymbol{C}$ the map $\boldsymbol{C}(\mathrm{C}, \mathrm{C}') \longrightarrow \boldsymbol{D}(\mathcal{F}\mathrm{C}, \mathcal{F}\mathrm{C}')$ is surjective (resp. injective). When $\mathcal{F}$ is both full and faithful it is called *fully faithful* for short (see [55, Remark 1.5.8, p.31]).

Definition 1.2. An *equivalence between categories* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ consists of functors $\mathcal{F} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ and $\mathcal{G} : \boldsymbol{D} \longrightarrow \boldsymbol{C}$ together with natural isomorphisms $\mathbf{1}_{\boldsymbol{C}} \cong \mathcal{G}\mathcal{F}$ and $\mathbf{1}_{\boldsymbol{C}} \cong \mathcal{F}\mathcal{G}$. Categories $\boldsymbol{C}$ and $\boldsymbol{D}$ are *equivalent* if there is an equivalence between them.[1]

Remark 1.2.1. Given a functor $\mathcal{F} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ and two objects C, C' of $\boldsymbol{C}$, if C, C' are isomorphic then $\mathcal{F}C$, $\mathcal{F}C'$ are isomorphic as well. If moreover $\mathcal{F}$ is fully faithful, then C, C' are isomorphic if and only if $\mathcal{F}C$, $\mathcal{F}C'$ are isomorphic.

Definition 1.3. A functor $\mathcal{F} : \boldsymbol{C} \longrightarrow \boldsymbol{D}$ is *essentially surjective on objects* if for every object D of $\boldsymbol{D}$ there is an object C of $\boldsymbol{C}$ such that D is isomorphic to $\mathcal{F}C$.

Theorem 1.4 below is well-known, we introduce it in order to clarify the meaning of Yoneda's lemma (see [55, Theorem 1.5.9, p.31]).

Theorem 1.4. *A functor defining an equivalence of categories is fully faithful and essentially surjective on objects. Assuming the axiom of choice, every functor with these properties defines an equivalence of categories.*

Definition 1.5. Let $\boldsymbol{C}$ be a category and C an object of $\boldsymbol{C}$. The *functor of points of* C, denoted by h_C, is the contravariant functor on $\boldsymbol{C}$ with values in $\boldsymbol{Set}$, the category of sets, that sends an object D of $\boldsymbol{C}$ to the set $\boldsymbol{C}(D, C)$, and an arrow $f \in \boldsymbol{C}(D, D')$ to the map

$$\begin{aligned} f^* \ : \ h_C(D') &\longrightarrow h_C(D) \\ g &\longmapsto g \circ f. \end{aligned}$$

Definition 1.6. Let $\boldsymbol{C}$ be a category. The *Yoneda embedding* is a covariant functor on $\boldsymbol{C}$ with values in the (non-necessarily locally small) category $\boldsymbol{Set}^{\boldsymbol{C}}$ of functors on $\boldsymbol{C}$ with values in $\boldsymbol{Set}$. Over objects, it is defined as

$$\begin{aligned} h \ : \ \boldsymbol{C} &\longrightarrow \boldsymbol{Set}^{\boldsymbol{C}} \\ C &\longmapsto h_C, \end{aligned}$$

and over arrows as sending $(f : C \longrightarrow C') \in \boldsymbol{C}(C, C')$ to the following natural transformation $h_f : h_C \longrightarrow h_{C'}$. Given an object D of $\boldsymbol{C}$, the map $h_f(D)$ is

$$\begin{aligned} f_* \ : \ h_C(D) &\longrightarrow h_{C'}(D) \\ g &\longmapsto f \circ g. \end{aligned}$$

The following two lemmas are well-known, see [55, Theorem 2.2.4, p.57] or [18, pp.13, 14].

Lemma 1.7 (Yoneda, weak version)**.** *Let* $\boldsymbol{C}$ *be a category. The Yoneda embedding* $h : \boldsymbol{C} \longrightarrow \boldsymbol{Set}^{\boldsymbol{C}}$ *is a fully faithful functor.*

Remark 1.7.1. By Remark 1.2.1 and weak Yoneda's lemma, two objects C, C' of a category $\boldsymbol{C}$ are isomorphic if and only if the functors h_C and h_D are isomorphic as well.

1 Two categories are equivalent when they are isomorphic as objects in the 2-category of categories, functors and natural transformations.

Lemma 1.8 (Yoneda). *Let $\mathcal{F} : \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ be a contravariant functor. Let* C *be an object of* $\boldsymbol{C}$ *and* ξ *an element of* $\mathcal{F}\,C$*. Given an object* D *of* $\boldsymbol{C}$*, the map*

$$\begin{aligned} \tau_\xi(D) \;:\; h_C(D) &\longrightarrow \mathcal{F}(D) \\ g &\longmapsto \mathcal{F}(g)(\xi) \end{aligned}$$

is natural on D*, that is it determines a natural transformation from* h_C *to* $\mathcal{F}$*. Moreover, the assignment* $\xi \longrightarrow \tau_\xi$ *determines one-to-one correspondence between* $\mathcal{F}(C)$ *and* $\boldsymbol{Set}^{\boldsymbol{C}}(h_C, \mathcal{F})$*.*

Definition 1.9. Let $\mathcal{F} : \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ be a contravariant functor on a category $\boldsymbol{C}$ with values in sets. A *representation of* $\mathcal{F}$ is a couple (C, ξ) with C an object of $\boldsymbol{C}$ and ξ an element of $\mathcal{F}\,C$ such that $\tau_\xi : h_C \longrightarrow \mathcal{F}\,C$ (see Lemma 1.8) is a natural isomorphism. The object C is called a *representing object*. The functor $\mathcal{F}$ is called *representable* when such a representation exists. When $\mathcal{F}$ corresponds to a parameter space problem, ξ is called a *universal family*.

For a category $\boldsymbol{C}$, the collection of all representable functors form a full subcategory of $\boldsymbol{Set}^{\boldsymbol{C}}$ and weak Yoneda's lemma establishes that $\boldsymbol{C}$ is equivalent to it.

Definition 1.10. Let $\boldsymbol{C}$ be a category and C an object of $\boldsymbol{C}$. The object C is *terminal* (resp. *initial*) in $\boldsymbol{C}$ if for every object D of $\boldsymbol{C}$ there is a unique arrow from D to C (resp. from C to D).

Given a category $\boldsymbol{C}$, if there are two terminal objects C and C' in $\boldsymbol{C}$, then there is a unique arrow f in $h_C(C')$ and a unique arrow g in $h_{C'}(C)$. The arrows $g \circ f$ and $f \circ g$ are the corresponding identity arrows of C and C' because these are the unique arrows in $h_C(C)$ and $h_{C'}(C')$. That is, when it exists, a terminal object is uniquely determined up to a unique isomorphism and, taking some liberty with the language, terminal objects are usually referred as *the* terminal object of a category.

Many times, we will define an object as the terminal object of a suitable category. In this cases, we will emphasise the previous observation with the sentence "by abstract nonsense such an object is uniquely determined up to a unique isomorphism".

Example 1.11. In the category $\boldsymbol{Set}$, every set with one element is a terminal object and between every pair of them there is a unique bijective map, so we represent *the* terminal object of $\boldsymbol{Set}$ by $\{*\}$. In $\boldsymbol{Set}$ the empty set is an initial object, there is a unique map from it to any set.

Example 1.12. In the category $\boldsymbol{Rings}$, the zero ring, a ring with one element where the additive and multiplicative neutral elements agree, is a terminal object.

In the category $\boldsymbol{Rings}$, the ring of integers $\mathbb{Z}$ is an initial object, given a ring A, a homomorphism $\mathbb{Z} \longrightarrow A$ is determined by the image of $1 \in \mathbb{Z}$, hence there is a unique homomorphism $\mathbb{Z} \longrightarrow A$.

Example 1.13. By previous examples, the empty scheme, whose ring of functions is the zero ring, is an initial object in the category of schemes.

And an immediate consequence is that Spec($\mathbb{Z}$) is a terminal object in the category of affine scheme, and in fact it is a terminal object in the category $\boldsymbol{Sch}$, we leave the details on gluing morphisms.

Definition 1.14. Let $\mathcal{F}: \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ be a contravariant functor on a category $\boldsymbol{C}$ with values in sets. The *category of elements of* $\mathcal{F}$, denoted by $\int \mathcal{F}$, is the category whose objects are couples (C, η) with C an object of $\boldsymbol{C}$ and η an element of $\mathcal{F}(C)$. Arrows $(C, \eta) \longrightarrow (C', \eta')$ in $\int \mathcal{F}$ are arrows $f \in \boldsymbol{C}(C, C')$ such that $\mathcal{F}(f)(\eta') = \eta$.

Equivalently, the category $\int \mathcal{F}$ may be defined as the comma category between functors $(h \downarrow \mathcal{F})$ or the opposite category to the comma category $(\mathbf{1} \downarrow \mathcal{F})$ (see [46, Chapter III] or [55, Exercise 1.3.vi, p.22 and §2.4, pp.66–72]).

Proposition 1.15. *Let $\mathcal{F}: \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ be a contravariant functor on a category $\boldsymbol{C}$ with values in sets, C an object of $\boldsymbol{C}$ and η an element of $\mathcal{F}(C)$. The couple (C, η) represents $\mathcal{F}$ if and only if it is the terminal object of $\int \mathcal{F}$.*

Proof sketch. Assume that (C, η) represents $\mathcal{F}$ via a natural isomorphism $\theta: \mathcal{F} \longrightarrow h_C$. Notice that $\eta = \theta(C)^{-1}(\mathbf{1}_C)$. Then, given an object (D, τ) of $\int \mathcal{F}$ there is a unique morphism $f \in h_C(D)$ such that $\theta(D)^{-1}(f) = \tau$ and this is the unique morphism in $\int \mathcal{F}$ from (D, τ) to (C, θ).

Assume that (C, η) is a terminal object in $\int \mathcal{F}$. That is, given an object D of $\boldsymbol{C}$ and an element $\tau \in \mathcal{F}(D)$ there is a unique morphism $f \in h_C(D)$ such that $\mathcal{F}(f)(\tau) = \eta$. So, a map $\theta(D): \mathcal{F}(D) \longrightarrow h_C(D)$ is defined as sending $\tau \in \mathcal{F}(D)$ to $f \in h_C(D)$, which is natural in D and in fact it is a natural isomorphism. □

Remark 1.15.1. The construction of the category $\int \mathcal{F}$ translates objects representing $\mathcal{F}$ into terminal objects. Given a category $\boldsymbol{C}$, there is a functor $\mathcal{F}_{\boldsymbol{C}}$ solving the inverse problem, an object of $\boldsymbol{C}$ is terminal if and only if it represents $\mathcal{F}_{\boldsymbol{C}}$. Namely, the functor $\mathcal{F}_{\boldsymbol{C}}: \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ sends every object to the terminal object of $\boldsymbol{Set}$ and every arrow to the identity (it can be seen as the constant diagram $\Delta_{\{*\}}: \boldsymbol{C} \longrightarrow \boldsymbol{Set}$, see Example 1.22).

In general, such constructions are not inverse to each other. But there is a case, with which we will work later (see Definitions 1.72 and 3.14), where they almost are. Let $\boldsymbol{C}$ be a subcategory of a category $\boldsymbol{D}$. Consider the following condition,

ቀ. if $D \longrightarrow C$ is an arrow in $\boldsymbol{D}$ and C is an object of $\boldsymbol{C}$, then D is also an object of $\boldsymbol{C}$, that is there are no arrows in $\boldsymbol{D}$ from objects outside $\boldsymbol{C}$ to objects of $\boldsymbol{C}$.

When $\boldsymbol{C}$, $\boldsymbol{D}$ satisfy ቀ, we may consider the characteristic (contravariant) functor $\mathcal{F}: \boldsymbol{D} \longrightarrow \boldsymbol{Set}$ of $\boldsymbol{C}$, that sends an object D of $\boldsymbol{D}$ to

$$\mathcal{F}D = \begin{cases} \{*\} & \text{if } D \text{ is an object of } \boldsymbol{C} \\ \emptyset & \text{otherwise.} \end{cases}$$

And, by ♆, it is well defined over arrows in the obvious way. Then, clearly $\int \mathcal{F} = \boldsymbol{C}$ and $\mathcal{F}_{\boldsymbol{C}} = \mathcal{F}|_{\boldsymbol{C}}$.

The following definition and its subsequent lemma provide a standard criterion for the representability of functors on the category of schemes $\boldsymbol{Sch}$.

Definition 1.16. Let $\mathcal{F}: \boldsymbol{Sch} \longrightarrow \boldsymbol{Set}$ be a contravariant functor on the category of schemes with values in sets.

1. A *subfunctor* $\mathcal{H} \subset \mathcal{F}$ is a rule that associates to every scheme T a subset $\mathcal{H}(T) \subset \mathcal{F}(T)$ such that the map $\mathcal{F}(f): \mathcal{F}(T) \longrightarrow \mathcal{F}(T')$ maps $\mathcal{H}(T)$ into $\mathcal{H}(T')$ for all morphisms $f: T' \longrightarrow T$.

2. Let $\mathcal{H} \subset \mathcal{F}$ be a subfunctor. The subfunctor $\mathcal{H} \subset \mathcal{F}$ is *representable by open* (resp. *closed*) (resp. *locally closed*) *embeddings* if for all pairs (T, ξ), where T is a scheme and $\xi \in \mathcal{F}(T)$ there is an open (resp. closed) (resp. locally closed) subscheme $U_\xi \subset T$ with the following universal property:

   $(*)$ A morphism $f : T' \longrightarrow T$ factors through U_ξ if and only if $f^*\xi \in \mathcal{H}(T')$.

Lemma 1.17. *Let $\mathcal{F}: \boldsymbol{Sch} \longrightarrow \boldsymbol{Set}$ be a contravariant functor on the category of schemes with values in sets. Let $\mathcal{H}$ be a subfunctor of $\mathcal{F}$ representable by open (resp. closed) (resp. locally closed) embeddings. If $\mathcal{F}$ is represented by a scheme X, then $\mathcal{H}$ is also representable and the representing scheme is an open (resp. closed) (resp. locally closed) subscheme of X.*

Proof. We just show the case that $\mathcal{H}$ is representable by open embeddings. Since we assume $\mathcal{F}$ representable, identifying $\mathcal{F}$ with h_X, we may assume that $\mathcal{H}$ is a subfunctor of h_X representable by open embeddings. Hence, for the couple $(X, \mathbf{1}_X)$, where $\mathbf{1}_X \in h_X(X)$, there is an open subscheme U (obviously unique) satisfying the following universal property: A morphism $f: T \longrightarrow X$ factorises through $U \hookrightarrow X$ if and only if $f^*\mathbf{1}_X = f \in \mathcal{H}(T)$. Now, just observe that if f factorises through $U \hookrightarrow X$ via a morphism $g_f: T \longrightarrow U$, then $g_f \in h_U(T)$ is unique because $U \hookrightarrow X$ is a monomorphism. Hence, there is an assignment $f \in \mathcal{H}T$ with $g_f \in h_U(T)$ which defines a natural transformation, and in fact a natural isomorphism. □

1.1.2 *Pullbacks and pushouts*

> Just when I thought I was out,
> THEY PULL ME BACK IN.
>
> –SILVIO DANTE imitates AL PACINO
> *The Soprano*

Pullbacks in the category of schemes and (its dual for affine schemes) pushouts in the category of commutative rings with unity are one of our basic tools to construct new morphisms. This section introduces the categorical notion of pullback and pushout, and proves some of their categorical properties.

Definition 1.18. Let $\mathcal{C}$ be a category and $\mathbf{D}$ a small category (that is, the objects of $\mathbf{D}$ form a set). A *diagram in $\mathcal{C}$ of shape* $\mathbf{D}$ (or simply a diagram) is a functor $\mathbb{D}\colon\mathbf{D}\longrightarrow\mathcal{C}$.

Given a diagram $\mathbb{D}\colon\mathbf{D}\longrightarrow\mathcal{C}$, the indexing category $\mathbf{D}$ is usually thought of as a formal category just shaping the diagram.

Remark 1.18.1. Diagrams in $\mathcal{C}$ with shape $\mathbf{D}$ form a category, it is simply the category $\mathcal{C}^{\mathbf{D}}$, which by [55, Remark 1.7.3, p.44] is locally small. That is, natural transformations between two diagrams form a set.

Example 1.19. Let $\mathbf{sq}$ denote the category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with four objects and five non-identity arrows, which respect compositions. A diagram of shape $\mathbf{sq}$ corresponds to the common notion of a square diagram that commutes. We will omit the diagonal arrow.

Example 1.20. Let $\mathbf{pb}$ denote the category

$$\bullet\longrightarrow\bullet\longleftarrow\bullet$$

with three objects and two non-identity arrows with common codomain. A diagram of shape $\mathbf{pb}$ corresponds to a pair of arrows with common codomain.

Example 1.21. Consider the category $\mathbf{pb}$ with labels $a\longrightarrow b\longleftarrow c$ and $1\longrightarrow 2\longleftarrow 3$. The objects of the category $\mathbf{pb}\times\mathbf{pb}$ are the ordered couples $(a,1)$, $(a,2)$, $(a;3)$, $(b,1)\ldots$ And an arrow between two couples corresponds to a pair of arrows in $\mathbf{pb}$ between the first and the second members of the couples. So, the shape of a diagram indexed by this category is as follows.

$$\begin{tikzcd} (a,1) \arrow[r] \arrow[d] & (b,1) \arrow[d] & (c,1) \arrow[l] \arrow[d] \\ (a,2) \arrow[r] & (b,2) & (c,2) \arrow[l] \\ (a,3) \arrow[r] \arrow[u] & (b,2) \arrow[u] & (c,3) \arrow[l] \arrow[u] \end{tikzcd}$$

Example 1.22. Given any category $\mathbf{D}$ and an object C of a category $\mathcal{C}$, the *constant diagram* $\Delta_C\colon\mathbf{D}\longrightarrow\mathcal{C}$ is the functor that sends every object to C and every arrow to the identity of C. Furthermore, every arrow $f\colon C\longrightarrow D$ in $\mathcal{C}$ determines trivially a natural transformation $\Delta_f\colon\Delta_C\longrightarrow\Delta_D$.

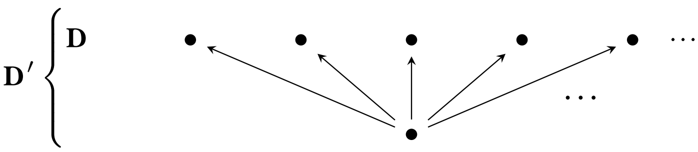
Definition 1.23. Let C be an object of a category $\boldsymbol{C}$. Let $\mathbb{D}\colon\mathbf{D}\longrightarrow\boldsymbol{C}$ be a diagram. A *cone* (resp. *cocone*) *with base* $\mathbf{D}$ *and vertex* C is a natural transformation $\eta\colon\Delta_C\longrightarrow\mathbb{D}$ (resp. $\eta\colon\mathbb{D}\longrightarrow\Delta_C$), where $\Delta_C\colon\mathbf{D}\longrightarrow\boldsymbol{C}$ is the constant diagram (see Example 1.22).

Example 1.24. Let $\boldsymbol{C}$ be a category and D an object of it. Given a diagram $\mathbb{D}\colon\mathbf{pb}\longrightarrow\boldsymbol{C}$ with image $A\longrightarrow B\longleftarrow C$, a cone η with base $\mathbb{D}$ and vertex D is given by three arrows $\eta_A\colon D\longrightarrow A$, $\eta_B\colon D\longrightarrow B$ and $\eta_C\colon D\longrightarrow C$ such that $D\longrightarrow B$ is equal to both compositions $D\longrightarrow A\longrightarrow B$ and $D\longrightarrow C\longrightarrow B$. Hence, the cone η is determined by the arrows η_A and η_B, which may be represented by the following diagram of shape $\mathbf{sq}$.

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & A \arrow[d] \\
C \arrow[r] & B
\end{tikzcd}$$

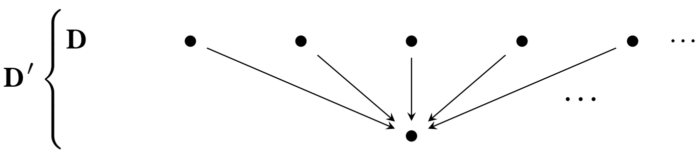
Example 1.25. Given a diagram $\mathbb{D}\colon\mathbf{D}\longrightarrow\boldsymbol{C}$, where $\mathbf{D}$ is the discrete category (a category with no non-identity arrows) over a set, which we also denote by $\mathbf{D}$, a cone η with base $\mathbb{D}$ is just a set of arrows of $\boldsymbol{C}$ with common domain. In this case, we call η a *discrete cone with vertex* C, where C is the vertex of η.

The cone η may also be seen as a diagram as follows. Consider a category $\mathbf{D}'$ constructed from $\mathbf{D}$ by adding formally an initial object, that is



So, the cone η is equivalent to the diagram $\mathbb{D}'\colon\mathbf{D}'\longrightarrow\boldsymbol{C}$ such that on objects it acts as $\mathbb{D}$ in $\mathbf{D}$ and it sends the new initial object to C. On arrows, it sends a non-identity arrow $\bullet\longrightarrow D$, where D is an object of $\mathbf{D}$, to $\eta_D\colon C\longrightarrow\mathbb{D}(D)$. Now it is even clearer that η corresponds a set of arrows $\{C\longrightarrow D_d\}_{d\in D}$.

Dually, a *discrete cocone with vertex* C is a cone with vertex C under a diagram with base the discrete category $\mathbf{D}$ over a set. Similarly, it may be seen as a diagram too, but the new category $\mathbf{D}'$ is now constructed from $\mathbf{D}$ by adding formally a terminal object, that is



Definition 1.26. Let $\mathbb{D}\colon\mathbf{D}\longrightarrow\boldsymbol{C}$ be a diagram. Then, there is a contravariant functor

$$\mathrm{Cone}(-,\mathbb{D})\colon\boldsymbol{C}\longrightarrow\boldsymbol{Set}$$

that sends an object C of $\boldsymbol{C}$ to the set (see Remark 1.18.1) of cones with base $\mathbb{D}$ and vertex C, and an arrow $f\colon C\longrightarrow D$ to the map $(\Delta_f)^*\colon\mathrm{Cone}(D,\mathbb{D})\longrightarrow\mathrm{Cone}(C,\mathbb{D})$. A *limit of* $\mathbb{D}$ is a representation of $\mathrm{Cone}(-,\mathbb{D})$.

Dually, there is a covariant functor

$$\mathcal{C}one(\mathbb{D},-)\colon \boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

that sends an object C of $\boldsymbol{C}$ to the set of cocones with base $\mathbb{D}$ and vertex C, and an arrow $f\colon C \longrightarrow D$ to the map $(\Delta_f)_*\colon \mathcal{C}one(D,\mathbb{D}) \longrightarrow \mathcal{C}one(C,\mathbb{D})$. A *colimit of* $\mathbb{D}$ is a representation of $\mathcal{C}one(\mathbb{D},-)$.

Given a diagram $\mathbb{D}\colon \boldsymbol{D} \longrightarrow \boldsymbol{C}$, observe that, by Proposition 1.15, a limit (resp. colimit) of $\mathbb{D}$ is a terminal (resp. initial) object of $\int\mathcal{C}one(-,\mathbb{D})$ (resp. $\int\mathcal{C}one(\mathbb{D},-)$).

So, when it exists, by abstract nonsense a limit or a colimit of a diagram $\mathbb{D}$ is uniquely determined up to a unique isomorphism.

Example 1.27. The limit (resp. colimit) of a diagram of shape **pb** is called a *pullback* (resp. *pushout*) (see [46, §4, Pullbacks, p.71]). All the results that follow about pullbacks have a co-version for pushouts reversing the arrows, we omit the details. A category $\boldsymbol{C}$ *admits pullbacks* if the pullback of every diagram of shape **pb** with values in $\boldsymbol{C}$ exists. Given a diagram $\mathbb{D}\colon \mathbf{pb} \longrightarrow \boldsymbol{C}$ with image $A \longrightarrow B \longleftarrow C$, when its pullback exists, the vertex is denoted by $A \times_B C$ and it is called again the pullback of A and C over B, or the fibre product of the pullback of A by $C \longrightarrow B$. The arrows $A \times_B C \longrightarrow A$ and $A \times_B C \longrightarrow C$ are called *projections*, and all the relevant arrows can be summarised in a diagram of shape **sq**

$$\begin{tikzcd}
A \times_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d] \\
C \arrow[r] & B,
\end{tikzcd}$$

which is called a *Cartesian square*. We usually emphasise a Cartesian square with a little corner inside it as in the previous diagram. Furthermore, a diagram $\mathbb{D}\colon \mathbf{D} \longrightarrow \boldsymbol{C}$ is called *Cartesian* if there is at least one functor $\mathcal{F}$: $\mathbf{sq} \longrightarrow \mathbf{D}$ injective on objects and, for all of them, the diagrams $\mathbb{D} \circ \mathcal{F}$ are Cartesian squares.

Consider a cone η with base $\mathbb{D}$ and vertex D (see Example 1.24). Since the pullback is the terminal object in the category $\int\mathcal{C}one(-,\mathbb{D})$, there is a unique arrow $D \longrightarrow A \times_B C$ (which we denote by $\eta_A \times_B \eta_C$) such that

$$(\Delta_{(\eta_S \times_B \eta_C)})^*\colon \mathcal{C}one(A \times_B C, \mathbb{D}) \longrightarrow \mathcal{C}one(D,\mathbb{D})$$

maps the pullback $A \times_B C$ to η.

Definition 1.28. Let $\boldsymbol{C}$ be a category admitting pullbacks and $f\colon A \longrightarrow B$ an arrow in $\boldsymbol{C}$. Via the identity arrow of A and f, there is a cone over the diagram $A \xrightarrow{f} B \xleftarrow{f} A$ with vertex A. We denote by $\Delta_{A/B}$ the arrow $\mathbf{1}_A \times_B \mathbf{1}_A$, which is called *the diagonal of* A *over* B.

Example 1.29. For every category $\boldsymbol{C}$ and every arrow $f\colon A \longrightarrow C$ of $\boldsymbol{C}$, the following diagram is Cartesian.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathbf{1}_A"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "\mathbf{1}_C"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

Lemma 1.30 (Transtivity of pullbacks, see [46, Exercise 8, p.72]). *Let $\boldsymbol{C}$ be a category admitting pullbacks. Let $A \longrightarrow B$ be an arrow of $\boldsymbol{C}$. Let $B \longrightarrow C \longleftarrow D$ be a diagram in $\boldsymbol{C}$ of shape* $\mathbf{pb}$. *Then, the pullback of the diagram $A \longrightarrow B \longleftarrow B \times_C D$ is isomorphic to the pullback of the diagram $A \longrightarrow C \longleftarrow D$ obtained by composition. In particular, given the following diagram,*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z'
\end{tikzcd}$$

if the right hand square is Cartesian, then the left hand square is Cartesian if and only if so is the big one obtained by composition.

Proof. Since we may consider the following diagram of shape $\mathbf{pb} \times \mathbf{pb}$,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] & B \arrow[l, "\mathbf{1}_B"'] \arrow[d] \\
C \arrow[r, "\mathbf{1}_C"] & C & C \arrow[l, "\mathbf{1}_C"'] \\
C \arrow[r, "\mathbf{1}_C"] \arrow[u, "\mathbf{1}_C"] & C \arrow[u, "\mathbf{1}_C"] & D \arrow[l] \arrow[u]
\end{tikzcd}$$

and limits commute with limits (see [55, Theorem 3.8.1, p.111]),

$$(A \times_C C) \times_{(B \times_C C)} (B \times_C D) \cong (A \times_B B) \times_{(C \times_C C)} (C \times_C D).$$

Hence, by Example 1.29, $A \times_B (B \times_C D) \cong A \times_C D$. $\square$

Lemma 1.31 (Magic diagram, see [61, Exercise 1.3.S, pp.36, 37]). *Let $\boldsymbol{C}$ be a category admitting pullbacks. Let $A \longrightarrow B$ be an arrow of $\boldsymbol{C}$. Let $C \longrightarrow B \longleftarrow D$ and $C \longrightarrow A \longleftarrow D$ be diagrams in $\boldsymbol{C}$ of shape* $\mathbf{pb}$. *The arrow $A \longrightarrow B$ determines an arrow $C \times_A D \longrightarrow C \times_B D$, and moreover, the following diagram is Cartesian.*

$$\begin{tikzcd}
C \times_A D \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \times_B D \arrow[d] \\
A \arrow[r, "\Delta_{A/B}"] & A \times_B A
\end{tikzcd}$$

Proof. Since we may consider the following diagram of shape $\mathbf{pb} \times \mathbf{pb}$,

$$\begin{tikzcd}
A \arrow[r, "\mathbf{1}_A"] \arrow[d, "\mathbf{1}_A"'] & A \arrow[d] & C \arrow[l] \arrow[d] \\
A \arrow[r] & B & B \arrow[l, "\mathbf{1}_B"'] \\
A \arrow[r, "\mathbf{1}_A"] \arrow[u, "\mathbf{1}_A"] & A \arrow[u] & D \arrow[l] \arrow[u]
\end{tikzcd}$$

and limits commute with limits (see [55, Theorem 3.8.1, p.111]),

$$(A \times_A A) \times_{(A \times_B A)} (C \times_B D) \cong (A \times_A C) \times_{(A \times_B B)} (A \times_A D).$$

Hence, by Example 1.29, $A \times_{(A \times_B A)} (C \times_B D) \cong C \times_A D$. $\square$

Lemma 1.32. *Let $\mathcal{C}$ be a category admitting pullbacks. Let $f\colon A \longrightarrow B$ be a monomorphism of $\mathcal{C}$ and $C \xrightarrow{g} B$ an arrow of $\mathcal{C}$. Consider the following Cartesian square.*

$$\begin{tikzcd} A \times_B C \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

The arrow p is also a monomorphism. Moreover, it is an isomorphism if and only if there is an arrow $C \longrightarrow A$ (obviously unique) whose composition with f is g.

Proof. Given two arrows $a, b\colon D \longrightarrow A \times_B C$ such that $pa = pb$, since f is a monomorphism, $qa = qb$. Setting $p' = pa = pb$ and $q' = qa = qb$, the arrows p' and q' define a cone with base $A \longrightarrow B \longleftarrow C$ and vertex D. Hence, by the uniqueness of $p' \times_B q'$, both arrows a and b are equal to it.

Now, if p is an isomorphism, the arrow $C \longrightarrow A$ is the composition qp^{-1}. If there is an arrow $h\colon C \longrightarrow A$ with $f = gh$, then the arrows h and $\mathbf{1}_C$ define a cone with base $A \longrightarrow B \longleftarrow C$ and vertex C. So, $p(h \times_D \mathbf{1}_C) = \mathbf{1}_C$ by definition, then $p(h \times_B \mathbf{1}_C)p = p$ and, since p is a monomorphism, $(h \times_B \mathbf{1}_C)p = \mathbf{1}_{A \times_B C}$. □

Definition 1.33. Given a category $\mathcal{C}$ and an object C of $\mathcal{C}$, *the slice category over* C (a particular case of a comma category), denoted by $\mathcal{C}_C$, has for objects arrows $X \longrightarrow C$ of $\mathcal{C}$ and for arrows (which are called C-arrows), say from $X \longrightarrow C$ to $Y \longrightarrow C$, arrows $X \longrightarrow Y$ of $\mathcal{C}$ such that the composition $X \longrightarrow Y \longrightarrow C$ agrees with $X \longrightarrow C$.

Given a category $\mathcal{C}$ and an arrow $g\colon C \longrightarrow D$ of $\mathcal{C}$, pullbacks give a functor $\mathcal{P}_g$, *the base change*, on the slice category $\mathcal{C}_D$ with values in the slice category $\mathcal{C}_C$. It sends $X \longrightarrow D$ to the projection $X \times_D C \longrightarrow C$, which is denoted by $X_C \longrightarrow C$ or simply X_C. And a D-arrow $f\colon X \longrightarrow Y$ is sent to $f_T\colon X_T \longrightarrow Y_T$, called the *base change* of f by $C \longrightarrow D$, defined by the following Cartesian diagram, recall Lemma 1.30.

$$\begin{tikzcd} X_C \arrow[r, "f_C"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y_C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r] & D \end{tikzcd}$$

1.1.3 *Grothendieck topologies*

This section aims to introduce the theory behind Corollary 1.45.1, a fundamental stone in all the forthcoming constructions. We refer to [47] and [60, Tag 022A] for a deeper treatment of the subject.

Definition 1.34. Fix a category $\mathcal{C}$ admitting pullbacks. A *Grothendieck topology* is a category $\mathcal{C}$ together with a collection $\mathrm{Cov}(\mathcal{C})$ of discrete cocones (see Example 1.25) with values in $\mathcal{C}$ (see Example 1.25), which are called *coverings*, satisfying the following conditions.

(1) If $D \longrightarrow C$ is an isomorphism, then the discrete cocone $\{D \longrightarrow C\}$ belongs to $\mathrm{Cov}(\boldsymbol{C})$.

(2) For every covering $\{D_i \longrightarrow C\}_i$ in $\mathrm{Cov}(\boldsymbol{C})$ and every arrow $E \longrightarrow C$ of $\boldsymbol{C}$, the set of projections $\{D_i \times_C E \longrightarrow E\}_i$, which form a discrete cocone, belongs to $\mathrm{Cov}(\boldsymbol{C})$.

(3) For every covering $\{D_i \longrightarrow C\}_i$ in $\mathrm{Cov}(\boldsymbol{C})$ and every collection on i of coverings $\{E_{i,j} \longrightarrow D_i\}_j$ in $\mathrm{Cov}(\boldsymbol{C})$, the discrete cocone $\{E_{i,j} \longrightarrow C\}_{i,j}$, obtained by composition, belongs to $\mathrm{Cov}(\boldsymbol{C})$.

Example 1.35 (The small classic topology). Let X be a topological space and let $\mathrm{Open}(X)$ denote the category of open subsets of X, where arrows $U \longrightarrow V$ stand for inclusions $U \subseteq V$. Then, $\mathrm{Open}(X)$ with the set of discrete cocones $\{U_i \longrightarrow U\}_i$ such that $U \subseteq \cup_i U_i$ form a Grothendieck topology.

In this case, given two arrows $V \longrightarrow U$ and $W \longrightarrow U$, the pullback $V \times_U W$ is the intersection $V \cap W$.

Example 1.36 (The global classic topology). The category of topological spaces with the collection of discrete cocones $\{f_i : U_i \longrightarrow X\}_i$ such that each f_i is an open embedding and $X \subseteq \cup_i f_i(U_i)$ form a Grothendieck topology.

Notice that we must consider open embeddings in general, not just inclusions of open subspaces; otherwise condition (1) of Definition 1.34 is not satisfied.

Example 1.37 (The small Zariski topology). It is the small classic topology over the underlying topological space of a scheme.

Example 1.38 (The global Zariski topology). Let S be a ground scheme. Then, $\boldsymbol{Sch}_S$ with the collection of families of S-morphisms $\{f_i : U_i \longrightarrow X\}_i$ such that each f_i is an open embedding and $X \subseteq \cup_i f_i(U_i)$ form a Grothendieck topology.

Given a covering $\{f_i : U_i \longrightarrow X\}$, the set $\{f_i(U_i)\}_i$ is an open cover of X in the classic sense.

Definition 1.39. Let S be a ground scheme and X an S-scheme. An *fpqc covering of* X (see [18, Definition 2.34, p.28] and [60, Tag 022B]) is a cocone of S-morphisms $\{\varphi_i : U_i \longrightarrow X\}_{i \in I}$ such that

1. every φ_i is a flat morphism and $X \subseteq \cup_{i \in I} \varphi_i(S_i)$; and

2. for every affine open subscheme U of X, there is a finite set K, a map $\kappa : K \longrightarrow I$, and affine open subschemes $V_{\kappa(k)}$ of $S_{\kappa(k)}$ for $k \in K$ such that $U = \cup_{k \in K} \varphi_{\kappa(k)}(V_{\kappa(k)})$.

The *fpqc topology* over the category $\boldsymbol{Sch}_S$ is the Grothendieck topology over $\boldsymbol{Sch}_S$ where the collection $\mathrm{Cov}(\boldsymbol{Sch}_S)$ of coverings is given by the fpqc coverings of any S-scheme (see [18, pp.27, 28])

We call a morphism $X \longrightarrow S$ an fpqc morphism if the set $\{X \longrightarrow S\}$ is an fpqc covering of S.

Remark 1.39.1. A morphism $f\colon X \to S$ is an fpqc morphism if and only if it is flat and, for every affine open subscheme U of S, there is a finite set of affine open subschemes V_i of X such that $U = \cup_i f(U_i)$. In particular, f is faithfully flat, that is flat and surjective.

For Example 1.41 below, we recall the following definition.

Definition 1.40. A morphism $X \to S$ is *quasi-compact* if the preimage of every affine open subscheme of S is a quasi-compact open subset of X (see [20, Proposition and Definition 10.1, p.242] or [60, Tag 01K2]).

Example 1.41. For a quasi-compact morphism $f\colon X \to S$, the preimage of every affine open subscheme U of S (which is quasi-compact) is covered by a finite set of affine open subscheme of X. Hence, if moreover f is faithfully flat, then f is a fpqc morphism.

In fact, the abbreviation fpqc stands for "fidèlement plat et quasi-compact", meaning faithfully flat and quasi-compact in French; which was the class of morphisms used in Grothendieck's original definition of the fpqc topology (which is slightly more restrictive than the nowadays standard Definition 1.39).

Example 1.42. If $S = \mathrm{Spec}(\Bbbk)$, then every morphism $f\colon X \to S$ is flat and the unique affine open subscheme of S, which is S itself, is covered by any affine open subscheme of X. Hence, every morphism $f\colon X \to S$ is an fpqc morphism.

Definition 1.43. Let $\boldsymbol{C}$ be a category admitting pullbacks. Let $\mathrm{Cov}(\boldsymbol{C})$ be a Grothendieck topology over $\boldsymbol{C}$. A *presheaf on $\boldsymbol{C}$* is a contravariant functor $\mathcal{F}\colon \boldsymbol{C} \to \boldsymbol{Set}$ and it is a *sheaf for the topology* $\mathrm{Cov}(\boldsymbol{C})$ if for every covering $\{D_i \to C\}_i$ in $\mathrm{Cov}(\boldsymbol{C})$ the following diagram is an equaliser.

$$\mathcal{F}C \longrightarrow \coprod_i \mathcal{F}D_i \rightrightarrows \coprod_{i,j} \mathcal{F}(D_i \times_C D_j)$$

Which means that for every $(s_i) \in \coprod_i \mathcal{F}D_i$ satisfying $p^*s_i = q^*s_j$ for all i, j, where $p\colon D_i \times_C D_j \to D_i$ and $q\colon D_i \times_C D_j \to D_j$ are the projections, there is a unique $s \in \mathcal{F}C$ whose pullback by $D_i \to C$ is s_i for all i.

For a presheaf $\mathcal{F}\colon \boldsymbol{C} \to \boldsymbol{Set}$, when there is no confusion, given an arrow $\iota\colon D_i \to C$ of a covering in $\mathrm{Cov}(\boldsymbol{C})$, the image of an element $s \in \mathcal{F}(C)$ by $\mathcal{F}(\iota)$ is denoted by $s|_{D_i}$.

The same definition applies for a functor with values in any other category $\boldsymbol{D}$, besides $\boldsymbol{Set}$, in this case it is called a *sheaf with values in $\boldsymbol{D}$ for the topology* $\mathrm{Cov}(\boldsymbol{C})$.

Example 1.44. Given a functor $\mathcal{F}\colon \boldsymbol{Sch} \to \boldsymbol{Set}$, it is a sheaf for the global Zariski topology (see Example 1.38) if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in \mathcal{F}(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in \mathcal{F}(T)$ such that $\xi_i = \xi|_{U_i}$ in $\mathcal{F}(U_i)$.

Theorem 1.45 below is a result on descent due to Grothendieck. For a proof, we refer to [18, Theorem 2.55, p.34] (or [60, Tag 03O3]). Grothendieck's original result is [23, B.1, Théorème 2. (190-19)], which only applies to the original definition of an fpqc cover (see paragraph below Example 1.41).

Theorem 1.45 (Grothendieck). *A representable functor on $\mathcal{S}ch_S$ is a sheaf in the fpqc topology.*

Corollary 1.45.1. *Let S be a ground scheme. Let $\pi\colon X \longrightarrow T$ and $f\colon X \longrightarrow Y$ be S-morphisms and consider the two projections $p, q\colon X \times_T X \longrightarrow X$. Assume that π is an fpqc morphism, then there is a morphism $g\colon T \longrightarrow Y$ (obviously unique) such that $f = g \circ \pi$ if and only if $f \circ p = f \circ q$.*

Proof. The following diagram is an equaliser.

$$h_{Y/S}\, T \longrightarrow h_{Y/S}\, X \underset{q^*}{\overset{p^*}{\rightrightarrows}} h_{Y/S}(X \times_S X). \qquad \square$$

1.2 THE SCHEME THEORETIC IMAGE

In this section, we review the scheme theoretic version of the image of a morphism, while we set the notation. But first as a warm up, we discuss a bit about the subtle difference between $\sigma\colon Z \hookrightarrow X$ being a closed embedding or a closed subscheme. Basically, when Z is a closed subscheme of X there is a unique natural embedding $\sigma\colon Z \hookrightarrow X$, but when $\sigma\colon Z \longrightarrow X$ is a closed embedding there may be many embeddings of Z into X, none of them more natural than another.

For affine schemes the difference is clear: a closed subscheme corresponds to the natural homomorphism $A \longrightarrow A/I$, where A is a ring and I an ideal of A, and a closed embedding is just a surjective homomorphism $A \longrightarrow B$ for some rings A, B.

Definition 1.46. Let $f\colon X \longrightarrow Y$ be a morphism of schemes. The *scheme theoretic image of* f (or schematic image for short) is a closed subscheme $\mathrm{Im}(f)$ of Y through which f factorises and satisfying the following universal property: If f factorises through a closed embedding $Z \hookrightarrow Y$, then $\mathrm{Im}(f) \hookrightarrow Y$ also factorises through it. We also call a diagram $X \longrightarrow \mathrm{Im}(f) \hookrightarrow Y$ a scheme theoretic image. Given an open subscheme U of X the *schematic closure of* U *in* X is the schematic image of the open embedding $U \hookrightarrow X$.

In addition, given a point $x \in X$, we denote by $\overline{x}$ the schematic image of the natural morphism $\mathrm{Spec}(\kappa(x)) \longrightarrow X$.

Remark 1.46.1. It is a standard result (see [20, Definition and Lemma 10.29, p.251], [22, I, Chapitre I, §9.5, p.176] or [60, Tag 01R5]) that the schematic image of any morphism f exists, by abstract nonsense it is uniquely determined up to a unique isomorphism, but since it is a closed subscheme it is in fact unique.

If f is quasi-compact, then the closed subscheme $\mathrm{Im}(f)$ of Y is defined by the quasi-coherent $\mathcal{O}_Y$-ideal $\ker(\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X)$ (see [20, Proposition 10.30, p.251], [22, I, Chapitre I, §9.5, p.176] or [60, Tag 01R5]).

Remark 1.46.2. Fix a scheme Y and a monomorphism $i\colon Z \longrightarrow Y$ (e.g. a closed or open embedding). Since being an isomorphism is a local property on the target, by Lemma 1.32, for a morphism $f\colon X \longrightarrow Y$, the property of factorising through i is local on the source.

Lemma 1.47. *Let $X \longrightarrow \overline{X} \hookrightarrow Y$ be a schematic image and $i\colon Z \hookrightarrow Y$ a closed subscheme. Then, the closed embedding $Z_X \hookrightarrow X$ is an isomorphism if and only if so is $Z_{\overline{X}} \hookrightarrow \overline{X}$.*

Proof. The closed embedding $Z_X \hookrightarrow X$ is the base change of $Z_{\overline{X}} \hookrightarrow \overline{X}$ by $X \longrightarrow \overline{X}$, hence if the latter is an isomorphism then so is the former. On the other hand, if $Z_X \hookrightarrow X$ is an isomorphism, via its inverse, the morphism $X \longrightarrow Y$ factorises through $Z \hookrightarrow Y$. Then, by its universal property, the closed embedding $\overline{X} \hookrightarrow Y$ also factorises through $Z \hookrightarrow Y$ and the claim follows from Lemma 1.32. □

Lemma 1.48 below is another standard result about schematic images (see [20, Lemma 14.6, p.424], [60, Tag 081I] or [22, IV$_2$, Chapitre IV, Proposition 2.3.2, p.14]).

Lemma 1.48. *Let S be a ground scheme and $S' \longrightarrow S$ a flat morphism. Let $f\colon X \longrightarrow Y$ be a quasi-compact morphism of S-schemes and $\overline{X}$ its schematic image. The schematic image of the base change $f'\colon X' \longrightarrow Y'$ of f by $S' \longrightarrow S$ is the fibre product $\overline{X} \times_S S'$.*

Proposition 1.49. *Let $\pi\colon X \longrightarrow Y$ be a separated morphism. Then, every section $\sigma\colon Y \longrightarrow X$ of π is a closed embedding.*

Proof. Consider the schemes X and Y as X-schemes via $\mathbf{1}_X$ and σ respectively, and as Y-schemes via π and $\pi \circ \sigma = \mathbf{1}_Y$. Then, by Lemma 1.31, the following diagram is Cartesian.

$$\begin{tikzcd} Y \arrow[r, "\sigma"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\mathbf{1}_X \times_Y (\sigma\circ\pi)"] \\ X \arrow[r, "\Delta_{X/Y}"] & X \times_Y X \end{tikzcd}$$

Since $\pi\colon X \longrightarrow Y$ is separated, $\Delta_{X/Y}$ is a closed embedding and then σ is a closed embedding as well. □

1.3 SCHEME THEORETIC CONSTANT MORPHISMS

This sections paraphrases the common set-theoretic notion of a constant map for morphisms of schemes.

Definition 1.50. Let S be a ground scheme. Let $p\colon X \longrightarrow Y$ and $f\colon X \longrightarrow W$ be S-morphisms. Consider the following Cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[d, hook, "\Delta_{W/S}"] \\ X \times_Y X \arrow[r, "f \times_Y f"] & W \times_S W \end{tikzcd}$$

where $\Delta_{W/S}$ is the diagonal. We say that the morphism f is *constant along the fibres of* p if the monomorphism $Z \hookrightarrow X \times_Y X$ is an isomorphism.

The standard (and maybe more intuitive) definition of a morphism $f\colon X \longrightarrow W$ being constant along the fibres of another morphism $p\colon X \longrightarrow Y$ is that the following diagram commutes.

$$\begin{tikzcd}
X \times_Y X \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & W
\end{tikzcd}$$

That is, the kernel, or equaliser, of the two morphisms $f \circ q_1, f \circ q_2$ is the whole scheme $X \times_Y X$, which, by Lemma 1.32, is equivalent to Definition 1.50 (see [27, Définition 1.4.2, p.34 and Proposition 1.4.10, p.37]).

When the underlying topological space of Y is just a point, we recover the set-theoretic notion of a constant map over the closed points of X. Indeed, for every pair of closed points x, x' of X, there is a point η of $X \times_Y X$ such that $q_1(\eta) = x$ and $q_2(\eta) = x'$, hence, if f is constant along the fibres of p,

$$f(x) = f \circ q_1(\eta) = f \circ q_2(\eta) = f(x').$$

Remark 1.50.1. From the second definition, it follows straightforwardly that, given an S-morphism $f'\colon W \longrightarrow W'$, if f is constant along the fibres of p, then so is $f' \circ f$. If furthermore f' is a monomorphism, then the converse also holds.

Remark 1.50.2. Given a morphism $g\colon X' \longrightarrow X$, since the following diagram is Cartesian,

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[d, hook, "\Delta_{W/S}"] \\
X' \times_W X' \arrow[r, "g \times_W g"] & X \times_W X \arrow[r, "f \times_W f"] & W \times_S W
\end{tikzcd}$$

if f is constant along the fibres of p, then $f \circ g$ is constant along the fibres of $p \circ g$.

Proposition 1.51. *Let S be a ground scheme. Let $p\colon X \longrightarrow Y$ and $f\colon X \longrightarrow W$ be S-morphisms. If p is an fpqc morphism (see Definition 1.39), then f is constant along the fibres of p if and only if there is an S-morphism $g\colon Y \longrightarrow W$ such that $f = g \circ p$. In this case, the morphism g is unique.*

Proof. The condition that f is constant along the fibres of p is just stetting that f belongs to the kernel of the two maps

$$q_1^*, q_2^*\colon h_{W/S}(X) \longrightarrow h_{W/S}(X \times_Y X),$$

where $q_1, q_2\colon X \times_Y X \longrightarrow X$ are the projections. Hence, the claim follows immediately form Corollary 1.45.1. □

1.3.1 *Naively constant morphisms*

A constant map $f\colon X \longrightarrow Y$ is commonly known as a map for which it exits a unique $y \in Y$ such that $f(x) = y$ for every $x \in X$. In this section, we explore the scheme-theoretic consequences of this set-theoretic definition.

Definition 1.52. We say that a morphism $f\colon X \longrightarrow Y$ is *naively constant through a point* y_0 *of* Y when $f(x) = y_0$ for all points x of X.

Theorem 1.53. *Let* $f\colon X \longrightarrow Y$ *be a morphism with* X *quasi-compact. The morphism* f *is naively constant through a point* y_0 *of* Y *if and only if it factors through a morphism* $Z \longrightarrow Y$ *where the underlying topological space of* Z *is a point and the underlying reduced subscheme* $\overline{Z}_{\mathrm{red}}$ *of the schematic closure* $\overline{Z}$ *of* $Z \longrightarrow Y$ *is equal to* $\overline{y_0}$*. Moreover, if* y_0 *is a closed point of* Y*, then the morphism* $Z \longrightarrow Y$ *is a closed embedding.*

Before proving Theorem 1.53, we characterise schemes whose underlying topological space is a point and we illustrate the two fundamental steps of our proof with two examples.

Proposition 1.54. *Given a scheme* Z*, its underlying topological space is a point if and only if* Z *is affine and the nilradical of its ring of functions* R *is a maximal ideal. In particular, the ring* R *is local.*

Proof. Consider a ring R for which $\mathrm{Nil}(R)$ is a maximal ideal. Since

$$\mathrm{Nil}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$$

and it is a maximal ideal of R, it is the unique prime ideal of R.

Conversely, if there is a unique $z \in Z$, then $\{z\} = Z$ is the only open neighbourhood of z, so it has to be affine, that is $Z \cong \mathrm{Spec}(R)$ for some ring R. Now, the prime ideal $\mathfrak{q}$ of R corresponding to z is the unique prime ideal of R, so $\mathfrak{q}$ is the nilradical of R and it is maximal. □

To illustrate the first step in the proof of Theorem 1.53 consider for every natural number n the n-th fat point scheme X_n, that is the spectrum of $R_n = \Bbbk[\varepsilon]/(\varepsilon^n)$. The natural homomorphism $\varphi_n\colon \Bbbk[x] \longrightarrow R_n$ gives a morphism $X_n \longrightarrow \mathbb{A}^1_{\Bbbk}$, which is naively constant and it factorises through X_n itself, notice that $\Bbbk[x]/\ker(\varphi_n) \cong R_n$ and that $\varphi_n^{-1}((\varepsilon)) = \sqrt{\ker(\varphi_n)}$. When we consider the scheme X defined as the finite disjoint union

$$X = \coprod_{n \in N} X_n,$$

for some finite set of natural numbers N, we may decompose the natural morphism $X \longrightarrow \mathbb{A}^1_{\Bbbk}$ into the affine morphisms φ_n for all $n \in N$, each one again factorising through $\Bbbk[x]/\ker(\varphi_n) \cong R_n$. But $X \longrightarrow \mathbb{A}^1_{\Bbbk}$ just factorises through

$$\Bbbk[x]/\Big(\bigcap_{n \in N} \ker(\varphi_n)\Big) = R_{\max(N)}.$$

The other required step is illustrated by the natural morphism from the generic point η of an irreducible plane curve to $\mathbb{A}^2_{\Bbbk}$. This morphism is naively constant and it factorises through η itself. But now, the corresponding homomorphism is $\varphi\colon \Bbbk[x,y] \longrightarrow \kappa(\eta)$ and $\Bbbk[x,y]/\ker(\varphi) \not\cong \kappa(\eta)$. In this case φ factorises through

$$(\Bbbk[x,y]/\ker(\varphi))_{\varphi^{-1}(\eta)}.$$

Proof of Theorem 1.53. Consider the affine case, that is $f\colon X \longrightarrow Y$ corresponds to a homomorphism $\varphi\colon A \longrightarrow B$ such that there is a unique prime ideal $\mathfrak{q}$ of A (the one corresponding to the point y_0 of Y) for which $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$ for every prime ideal $\mathfrak{p} \subseteq B$. Set

$$R = (A/\ker(\varphi))_{\mathfrak{q}}.$$

Let us check that $\varphi\colon A \longrightarrow B$ factorises naturally through R, that is (by the universal property of localisations) if $a \in A \setminus \mathfrak{q}$, then the image $\varphi(a) \in B$ is invertible. We show the contrapositive, so consider $a \in A$ for which $\varphi(a) \in B$ is non-invertible, then $\varphi(a)$ belongs to some prime ideal $\mathfrak{p}$ of B and $a \in \varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$.

Now, we show that R satisfies the condition of Proposition 1.54 and then its spectrum is a point. Since radicals commute with preimages,

$$\sqrt{\ker(\varphi)} = \varphi^{-1}(\mathrm{Nil}(B)).$$

Now, the ideal $\mathrm{Nil}(B)$ of B is the intersection of all the primes ideal of B hence, since intersections commute with preimages,

$$\sqrt{\ker(\varphi)} = \bigcap_{\mathfrak{p} \in X} \varphi^{-1}(\mathfrak{p}) = \mathfrak{q}.$$

In particular, $\mathfrak{q}$ is the unique prime ideal of $A_{\mathfrak{q}}$ containing $\ker(\varphi)$ and then R is a local ring whose nilradical is the maximal ideal.

For the general case, fix an affine open cover $\{U_i\}_{i \in I}$ of X with $U_i \cong \mathrm{Spec}(B_i)$ for some rings B_i. Since X is assumed quasi-compact, we assume the set I finite. Fix an affine open neighbourhood V of $y_0 \in Y$, say $V \cong \mathrm{Spec}(A)$ for a ring A, and denote by $\mathfrak{q}$ the prime ideal of A corresponding to y_0.

By assumption, the set $f(X)$ is $\{y_0\}$ which is a subset of V, so f factorises through the open embedding $V \hookrightarrow Y$ and so do all the restrictions $f|_{U_i}$. Denote by $\varphi_i\colon A \longrightarrow B_i$ the homomorphism corresponding to $f|_{U_i}\colon U_i \longrightarrow V$ and by $\mathfrak{a}_i$ its kernel. Now the desired scheme is the spectrum of

$$R = \left(\frac{A}{\bigcap_i \mathfrak{a}_i}\right)_{\mathfrak{q}}.$$

For every $i \in I$, by the affine case there is a morphism $(A/\mathfrak{a}_i)_{\mathfrak{q}} \longrightarrow B_i$ which extends to a morphism $\varphi_i\colon R \longrightarrow B_i$ through the natural morphism $R \longrightarrow (A/\mathfrak{a}_i)_{\mathfrak{q}}$ given by the inclusion $\bigcap_j \mathfrak{a}_j \hookrightarrow \mathfrak{a}_i$. For every $i \in I$, also by the affine case, $\sqrt{\mathfrak{a}_i} = \mathfrak{q}$. So, since radicals commute with finite products and I is finite,

$$\sqrt{\bigcap_{i \in I} \mathfrak{a}_i} = \bigcap_{i \in I} \sqrt{\mathfrak{a}_i} = \mathfrak{q}.$$

Hence, the spectrum of R is a point by Proposition 1.54.

Let us see that $f: X \longrightarrow Y$ factorises through $\operatorname{Spec}(R) \longrightarrow Y$ (here $\operatorname{Spec}(R) \longrightarrow Y$ is the composition $\operatorname{Spec}(R) \longrightarrow \operatorname{Spec}(A) \hookrightarrow Y$). We just need to check that the morphisms $\alpha_i : R \longrightarrow B_i$ define a morphism $X \longrightarrow Z$, that is they agree on overlaps. Fix two elements U_i and U_j of the cover $\{U_i\}_i$ with non-empty overlap. Let U be an affine open subscheme of $U_i \cap U_j$, say $U \cong \operatorname{Spec}(C)$ for some ring C, and consider the homomorphisms $c_i : B_i \longrightarrow C$ and $c_j : B_j \longrightarrow C$ corresponding respectively to the open embeddings $U \hookrightarrow U_i$ and $U \hookrightarrow U_j$. So, by construction, the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"'] & B_i \arrow[d, "c_i"] \\ B_j \arrow[r, "c_j"'] & C \end{tikzcd}$$

Now, it is straightforward to see that the following diagram also commutes.

$$\begin{tikzcd} A/(\cap_i \mathfrak{a}_i) \arrow[r] \arrow[d] & B_i \arrow[d] \\ B_j \arrow[r] & C \end{tikzcd}$$

and, since the localisation $A/(\cap_i \mathfrak{a}_i) \longrightarrow R$ is an epimorphism, $c_i \circ \alpha_i = c_j \circ \alpha_j$.

To show that the subschemes $\overline{Z}_{\mathrm{red}}$ and $\overline{y_0}$ of Y are equal we just need to show that

$$\sqrt{\ker(A \longrightarrow R)} = \mathfrak{q}.$$

In fact, $\ker(A \longrightarrow R) = \cap_i \mathfrak{a}_i$ because $A/\cap_i \mathfrak{a}_i \longrightarrow R$ is injective, let us check it. Observe that for every $i \in I$ the following diagram commutes.

$$\begin{tikzcd} A/\mathfrak{a}_i \arrow[d] \arrow[dr, hook] & \\ (A/\mathfrak{a}_i)_{\mathfrak{q}} \arrow[r] & B_i \end{tikzcd}$$

So, $A/\mathfrak{a}_i \longrightarrow (A/\mathfrak{a}_i)_{\mathfrak{q}}$ is injective for every i. For every $i \in I$, the following diagram also commutes.

$$\begin{tikzcd} A/\cap_i \mathfrak{a}_i \arrow[r] \arrow[d] & A/\mathfrak{a}_i \arrow[d] \\ R \arrow[r] & (A/\mathfrak{a}_i)_{\mathfrak{q}} \end{tikzcd}$$

Then, $\ker(A/\cap_i \mathfrak{a}_i \longrightarrow R) \subseteq \cap_i \ker(A/\cap_j \mathfrak{a}_j \longrightarrow A/\mathfrak{a}_i) = 0$.

Finally, if y_0 is a closed point, following with the same notation, the ideal $\mathfrak{q}$ of A is maximal and every ideal $\mathfrak{a}_i$ is $\mathfrak{q}$-primary. So, the nilradical of the ring $A/(\cap_i \mathfrak{a}_i)$ is a maximal ideal. $\square$

1.4 HILBERT SCHEMES

In sum, Grothendieck's method of representable functors is like Descartes' method of coordinate axes: simple, yet powerful. Here is one hallmark of genius!

—STEVEN L. KLEIMAN
The Picard scheme

Hilbert schemes are the parameter space for flat families of closed subschemes of quasiprojective schemes. The basic theory was developed by Grothendieck in [25]. Later, Altman and Kleiman, in [2], carried out in detail Grothendieck's exposition, which leads them to similar existence results with slightly different assumptions. For an alternative exposition, aiming to be more accessible, see [18, Chapter 5, pp.107–158]. For a clear exposition in the case of surfaces, see [48]. For another approach to Hilbert schemes replacing finiteness assumptions by ones more combinatorial and aiming for explicit equations see [29], which covers other approaches which also aim for explicit equations as [35] and [21].

The existence of our principal constructions, the blow up split section family and the universal scheme of ordered clusters of sections, relies on the existence of Hilbert schemes. In fact, we merely realise these constructions as locally closed subschemes of suitable Hilbert schemes.

In this section, we quickly review the basic definitions and existence results for Hilbert schemes.

Definition 1.55. We call a morphism $X \longrightarrow S$ *projective* (resp. *quasiprojective*) if it is finitely presented and there is a locally free $\mathcal{O}_S$-module E of constant finite rank together with a closed embedding (resp. a locally closed embedding) $X \hookrightarrow \mathbb{P}(E)$ over S.[1]

Definition 1.56. Let S be a ground scheme. Let X be a quasiprojective S-scheme. The *Hilbert functor of* X is the functor on $\boldsymbol{Sch}_S$ with values in $\boldsymbol{Set}$, denoted by $\mathcal{H}ilb_{X/S}$, which sends an S-scheme T to the set

$$\mathcal{H}ilb_{X/S}(T) = \{Z \text{ closed subscheme } X_T \text{ with } Z \longrightarrow T \text{ is proper and flat}\}.$$

Let $\mathbb{k}$ be a field, let X be a projective scheme over $\mathbb{k}$, and let $\mathcal{F}$ be a coherent sheaf on X whose support is proper over $\mathbb{k}$. For each $n > 0$, define

$$\xi(\mathcal{F}(n)) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{k}} H^i(X, \mathcal{F}(n)).$$

1 There are distinct notions of a (quasi)projective morphism (see the discussions in [2, p.52] and [18, §5.5.1, pp.126, 127]). For example, Definition 1.55 is equal to [2, Definition 2.1, p.63], where they call such schemes *strongly (quasi)projective*. We introduce this class of morphisms as finiteness conditions required for the existence of the Hilbert scheme, but all our existence results remain true assuming that the required Hilbert schemes exist.

Then, there is a polynomial $\Phi \in \mathbb{Q}[z]$, called the *Hilbert polynomial of* $\mathcal{F}$, such that $\Phi(n) = \xi(\mathcal{F}(n))$ for all $n > 0$. Given a closed subscheme Z of X, the *Hilbert polynomial of* Z is the Hilbert polynomial of the sheaf of $\mathcal{O}_X$-ideals defining Z. The Hilbert polynomial is a numerical invariant, which encodes a lot of information of the sheaf $\mathcal{F}$. For example, its degree is equal to the dimension of the support of $\mathcal{F}$. It enjoys many useful properties, but here we just review, with no proofs, the ones we are interested in. For a detailed treatment we refer to [22, III_1, Chapitre III, §2.5, pp.109–111 and III_2, Chapitre III, §7.9, pp.76–80] and [25, §2, pp.253–258].

Proposition 1.57. *Let $\Bbbk$ be a field, let X be a projective scheme over $\Bbbk$, and let $\mathcal{F}$ be a coherent sheaf on X whose support is proper over $\Bbbk$. Let $\Bbbk \subseteq K$ be a base field extension. Then, the Hilbert polynomial of the extended sheaf $\mathcal{F}_K$ on X_K is equal to the Hilbert polynomial of $\mathcal{F}$ on X.*

Proposition 1.58. *Let S be a ground scheme. Let $f\colon X \longrightarrow Y$ be a flat and projective S-morphism. Then, the Hilbert polynomial of the fibres of f is locally constant on the points of Y.*

Theorem 1.59 below is proved by Altman and Kleiman in [2, Theorem 2.6, p.66], where, by means of stronger notions of projectivity, they remove the Noetherian assumptions from the original result given by Grothendieck in [25, §3, pp.258–266 (Théorème 3.1, p.259)], see discussion in [2, p.52]. For an alternative exposition under Noetherian assumptions see [18, Chapter 5, pp.107–158].

Theorem 1.59 (Grothendieck). *Let S be a ground scheme. Let X be a projective (resp. quasiprojective) S-scheme. Then, the functor $\mathcal{H}ilb_{X/S}$ is representable and the representing S-scheme $\mathrm{Hilb}_{X/S}$ is at most a countable disjoint union of projective (resp. quasiprojective) S-schemes. Furthermore, if S is locally Noetherian, then so is $\mathrm{Hilb}_{X/S}$.*

Such a decomposition of the scheme $\mathrm{Hilb}_{X/S}$ is obtained as follows. For every polynomial $\Phi \in \mathbb{Q}[z]$, consider the subfunctor $\mathcal{H}ilb^{\Phi}_{X/S}$ of $\mathcal{H}ilb_{X/S}$, which sends an S-scheme T to the set

$$\{Z \in \mathcal{H}ilb_{X/S}(T) : \text{the Hilbert polynomial of the fibres of } Z \longrightarrow T \text{ is } \Phi\}.$$

By Proposition 1.58, the functor $\mathcal{H}ilb_{X/S}$ is representable if and only if all the functors $\mathcal{H}ilb^{\Phi}_{X/S}$ are representable, and, when they are representable, the S-scheme representing $\mathcal{H}ilb_{X/S}$ is isomorphic to the disjoint union of the schemes representing $\mathcal{H}ilb^{\Phi}_{X/S}$.[1]

1.5 SECTIONS IN FAMILY

We review the universal section family, a scheme parametrising all sections of a given morphism. We also state its main existence results as an open subscheme of a suitable Hilbert scheme.

1 Notice that, in general, given two S-schemes X and Y, the functors $h_{(X \sqcup Y)/S}$ and $h_{X/S} \sqcup h_{Y/S}$ are *not* isomorphic.

Definition 1.60. Let S be a ground scheme. Let $\pi\colon X\longrightarrow Y$ be an S-morphism. Let T be an S-scheme. A T*-family of sections over* π is a section $\sigma\colon Y_T\hookrightarrow X_T$ of the base change $\pi_T\colon X_T\longrightarrow Y_T$ of π by $T\longrightarrow S$, that is an element of $\boldsymbol{Sch}_{Y_T}(Y_T,X_T)$.

Let $\mathfrak{X}$ be an S-scheme and ψ an $\mathfrak{X}$-family of sections over π. The couple $(\mathfrak{X},\psi)$ is a *universal section family of* π (or Usf for short) if it satisfies the following universal property: For every S-scheme T and every T-family of sections σ over π, there is a unique S-morphism $f\colon T\longrightarrow\mathfrak{X}$ such that the following diagram is Cartesian,

$$\begin{tikzcd} Y_T \arrow[r, hook, "\sigma"] \arrow[d, "f_Y"'] \arrow[dr, phantom, "\ulcorner", very near start] & X_T \arrow[r, "\pi_T"] \arrow[d, "f_X"] \arrow[dr, phantom, "\ulcorner", very near start] & Y_T \arrow[d, "f_Y"] \\ Y_{\mathfrak{X}} \arrow[r, hook, "\psi"] & X_{\mathfrak{X}} \arrow[r, "\pi_{\mathfrak{X}}"] & Y_{\mathfrak{X}} \end{tikzcd} \tag{1.5.1}$$

or equivalently, such that $f_X\circ\sigma=\psi\circ f_Y$.

For explicit examples of universal section family see Examples 5.40 and 5.42.

Remark 1.60.1. The collection of families of sections over π form a category $\boldsymbol{S}_\pi$, where objects are pairs (T,σ) with T a S-scheme and σ a T-family of sections over π. An arrow, from (T,σ) to (T',σ'), is a morphism $f\colon T\longrightarrow T'$ such that $f_X\circ\sigma=f_Y\circ\sigma'$, or equivalently, such that σ is the base change of σ' by f.

If a Usf of an S-morphism $\pi\colon X\longrightarrow Y$ exists, by abstract nonsense, it is uniquely determined up to a unique isomorphism

Let S be a ground scheme. Given an S-morphism $\pi\colon X\longrightarrow Y$, consider the contravariant functor $\mathcal{S}ect_\pi\colon\boldsymbol{Sch}_S\longrightarrow\boldsymbol{Set}$ corresponding to the parameter space problem of sections of π, defined as follows. For every S-scheme T, set

$$\mathcal{S}ect_\pi T=\{\text{sections of }\pi_T\colon X_T\longrightarrow Y_T\}=\boldsymbol{Sch}_{Y_T}(Y_T,X_T),$$

and for every S-morphism $f\colon T'\longrightarrow T$, the map $\mathcal{S}ect_\pi f\colon\mathcal{S}ect_\pi T\longrightarrow\mathcal{S}ect_\pi T'$ sends a T-family of sections $\sigma\colon Y_T\hookrightarrow X_T$ over π to its base change $\sigma_{T'}\colon Y_{T'}\hookrightarrow X_{T'}$ by f, which is a T'-family of sections over π (see [24, II, C, **n.**2, pp.380, 381, le foncteur "ensemble des sections"]).

Proposition 1.61. *Let* S *be a ground scheme. Let* $\pi\colon X\longrightarrow Y$ *be an* S*-scheme. Let* $\mathfrak{X}$ *be an* S*-scheme and* ψ *a* $\mathfrak{X}$*-family of sections over* π*, that is* $\psi\in\mathcal{S}ect_\pi\mathfrak{X}$*. The couple* $(\mathfrak{X},\psi)$ *represents the functor* $\mathcal{S}ect_\pi$ *if and only if it is the Usf of* π*.*

Proof. By construction, $\boldsymbol{S}_\pi$ is the category of elements of $\mathcal{S}ect_\pi$, then the claim follows by Proposition 1.15. □

Let S be a ground scheme, $\pi\colon X\longrightarrow Y$ an S-morphism and T an S-scheme. Given a T-family of sections $\sigma\colon Y_T\hookrightarrow X_T$ over π, notice that for every S-point $s\colon S\longrightarrow T$ of T, the base change $\sigma_s\colon Y\hookrightarrow X$ of σ by s, that is $\sigma_s=\mathcal{S}ect_\pi(s)(\sigma)$, can be identified with a section of π.

Corollary 1.61.1. *Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-scheme. If the Usf $(\mathfrak{X}, \psi)$ of π exists, then the map $h_{\mathfrak{X}}(S) \longrightarrow \boldsymbol{Sch}_Y(Y, X)$ sending an S-point $s\colon S \longrightarrow \mathfrak{X}$ of $\mathfrak{X}$ to the section of π corresponding to ψ_s determines a one-to-one correspondence.*

Proposition 1.62. *Let S be a locally Noetherian ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-scheme with X quasiprojective over S and Y proper and flat over S. Then, the functor $\mathcal{S}ect_\pi$ is representable in the category of locally Noetherian schemes and it is represented by an open subscheme of $\mathrm{Hilb}_{X/S}$. In particular, the representing scheme is at most a countable disjoint union of quasiprojective schemes.*

In Section 3.1.2, after introducing the functor $\mathcal{I}so$, we show how representability of $\mathcal{I}so$ can be used to prove Proposition 1.62, see proof after Remark 3.16.2.

Remark 1.62.1. A natural question now is when $\mathcal{S}ect_\pi \hookrightarrow \mathcal{H}ilb_{X/S}$ is also representable by closed embeddings, that is, when the scheme representing $\mathcal{S}ect_\pi$ is projective.

By definition, it is needed that, for every locally Noetherian S-scheme T and every element $Z \in \mathcal{H}ilb_{X/S}T$, the open subscheme U_Z of T to be also a closed subscheme. By [60, Tag 05PF], that would be the case if the morphism of $\mathcal{O}_{Y_T}$-modules $\mathcal{O}_{Y_T} \longrightarrow (i_T \circ \pi_T)_* \mathcal{O}_Z$ is surjective, but this almost never happens.

For example, if π is affine, then $(\pi_T)_*$ is exact, in particular right exact and $(\pi_T)_*\mathcal{O}_X \longrightarrow (i_T \circ \pi_T)_*\mathcal{O}_Z$ is surjective, but we expect π to be surjective and $\mathcal{O}_{Y_T} \longrightarrow (\pi_T)_*\mathcal{O}_X$ injective.

A trivial case is when π is an isomorphism, then the scheme representing $\mathcal{S}ect_\pi$ is a point corresponding to its inverse (its unique section), or also corresponding to the connected component of the Hilbert scheme $\mathrm{Hilb}_{X/S}$ parametrising the whole scheme X as a subscheme of itself.

Proposition 1.62 can be easily extended to the case when the ambient scheme X is at most a countable disjoint union of quasiprojective schemes.

Proposition 1.63. *Let S be a locally Noetherian ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-scheme with X at most a countable disjoint union of quasiprojective schemes over S and Y proper and flat over S. Then, the functor $\mathcal{S}ect_\pi$ is representable and the representing scheme is locally Noetherian and at most a countable disjoint union of quasiprojective schemes.*

Proof. Fix a decomposition $\bigsqcup_{i \in I} X_i$ of X into a finite or countable disjoint union of quasiprojective schemes. For every $i \in I$, by Proposition 1.62, the Usf $(\mathfrak{X}_i, \psi_i)$ of $X_i \longrightarrow Y$ exists and $\mathfrak{X}_i$ is at most a countable disjoint union of quasiprojective schemes. So, $\mathfrak{X} = \bigsqcup_{i \in I} \mathfrak{X}_i$ is at most a countable disjoint union of quasiprojective schemes and, setting ψ as the composition of $\bigsqcup_{i \in I} \psi_i$ with the natural isomorphism $Y_{\mathfrak{X}} \longrightarrow \bigsqcup_{i \in I} Y_{\mathfrak{X}_i}$ (see [20, Exercise 4.2, p.115]), the couple $(\mathfrak{X}, \psi)$ is the Usf of π. □

1.5.1 *Elementary constructions*

In this section, we explore how the universal section family behaves under some elementary transformations, as base changes or pullbacks. It is mostly a warm up for Section 5.3.

Proposition 1.64. *Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-morphism. Let $T \longrightarrow S$ be a morphism. Assume that the universal section family $(\mathfrak{X}, \psi)$ of π exists. Then the universal section family of the T-morphism $\pi_T\colon X_T \longrightarrow Y_T$ is $(\mathfrak{X}_T, \psi_T)$.*

Proof. Given a T-scheme $T' \longrightarrow T$, clearly the image of the T-scheme $T' \longrightarrow T$ by the functor

$$\mathcal{S}ect_{\pi_T}\colon \boldsymbol{Sch}_T \longrightarrow \boldsymbol{Set}$$

and the image of the S-scheme $T' \longrightarrow T \longrightarrow S$ by the functor

$$\mathcal{S}ect_{\pi}\colon \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$$

agree. Finally, by the universal property of pullbacks there is an isomorphism

$$\boldsymbol{Sch}_S(T', \mathfrak{X}) \cong \boldsymbol{Sch}_T(T', \mathfrak{X}_T),$$

natural on T'. Hence, the scheme $\mathfrak{X}_T$ represents the functor $\mathcal{S}ect_{\pi_T}$. □

Proposition 1.65. *Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ and $\pi'\colon X' \longrightarrow Y$ be S-morphisms. Assume that the universal section families $(\mathfrak{X}, \psi)$ and $(\mathfrak{X}', \psi')$ of π and π' exist. Then, the universal section family of the S-morphism $\overline{\pi}$: $X \times_Y X' \longrightarrow Y$ is $(\mathfrak{X} \times_S \mathfrak{X}', \psi \times_S \psi')$.*

Proof. Given an S-scheme $T \longrightarrow S$, composition with the projections $X \times_Y X' \longrightarrow X$ and $X \times_S X' \longrightarrow X'$ give a map

$$\mathcal{S}ect_{\overline{\pi}}(T) \longrightarrow \mathcal{S}ect_{\pi}(T) \times \mathcal{S}ect_{\pi'}(T)$$

natural on T. Moreover, given to sections $\sigma\colon Y_T \longrightarrow X_T$ and $\sigma'\colon Y_T \longrightarrow X'_T$, since $\pi_T \circ \sigma = \mathbf{1}_{Y_T} = \pi'_T \circ \sigma'$, they determine a unique section $\sigma \times \sigma'\colon Y_T \longrightarrow X \times_Y X'$. That is, the previous map is bijective.

Finally, by Yoneda's lemma (see Lemma 1.7) there is an isomorphism

$$\boldsymbol{Sch}_S(T, \mathfrak{X}) \times \boldsymbol{Sch}_S(T, \mathfrak{X}') \cong \boldsymbol{Sch}_S(T, \mathfrak{X} \times_s \mathfrak{X}')$$

natural on T. □

Proposition 1.66. *Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ and $\pi'\colon X' \longrightarrow Y'$ be S-morphisms. Assume that the universal section families $(\mathfrak{X}, \psi)$ and $(\mathfrak{X}', \psi')$ of π and π' exist. Then, the universal section family of $\tilde{\pi}\colon X \times_s X' \longrightarrow Y \times_s Y'$ is $(\mathfrak{X}_{Y'} \times_S \mathfrak{X}'_Y, \psi_{Y'} \times_S \psi'_Y)$.*

Proof. By the following Cartesian diagram, it is an immediate consequence of Propositions 1.64 and 1.65.

$$\begin{tikzcd}
X \times_S X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X'_Y \arrow[r] \arrow[d, "\pi'_Y"] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d, "\pi'"] \\
X_{Y'} \arrow[r, "\pi_{Y'}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \times_S Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d] \\
X \arrow[r, "\pi"] & Y \arrow[r] & S
\end{tikzcd}$$

□

Lemma 1.67. *Let S be a ground scheme. Let $X \longrightarrow Y$ be a S-morphism and T an S-scheme. Consider the following Cartesian diagram.*

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d] & Y_T \arrow[r] \arrow[d] & T \arrow[d] \\
X \arrow[r, "\pi"] & Y \arrow[r] & S
\end{tikzcd}$$

Then, there is a bijective map from $\boldsymbol{Sch}_Y(Y_T, X)$ to $\boldsymbol{Sch}_{Y_T}(Y_T, X_T)$ natural on T.

Proof. Let $\sigma \in \boldsymbol{Sch}_{Y_T}(Y_T, X_T)$ and $f \in \boldsymbol{Sch}_Y(Y_T, X)$. Consider the following diagram.

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d] & Y_T \arrow[r] \arrow[l, bend right, dashed, "\sigma"'] \arrow[dl, dashed, "f"'] \arrow[d] & T \arrow[d] \\
X \arrow[r, "\pi"] & Y \arrow[r] & S
\end{tikzcd}$$

The composition of σ with the projection $X_T \longrightarrow X$ belongs to $\boldsymbol{Sch}_Y(Y_T, X)$. Since, $f \in \boldsymbol{Sch}_Y(Y_T, X)$, the product of f with the $\mathbf{1}_{Y_T}$ is well defined and it is an element of $\boldsymbol{Sch}_{Y_T}(Y_T, X_T)$. Clearly, these two constructions are natural on $T \longrightarrow S$ and mutually inverse. □

Proposition 1.68. *Let S be a ground scheme. Let $g\colon Y \longrightarrow S$ be a morphism. Let $\mathcal{F}\colon \boldsymbol{Sch}_Y \longrightarrow \boldsymbol{Set}$ be a representable functor represented by a pair (X, η), where X is a Y-scheme and η a natural isomorphism between $\mathcal{F}$ and h_X. Denote by $\pi\colon X \longrightarrow Y$ the structure morphism, which is an S-morphism. Then, there is a natural isomorphism*

$$\mathcal{F} \circ \mathcal{P}_g \cong \mathcal{S}ect_\pi,$$

where $\mathcal{P}_g$ is the base change functor (see Definition 1.33).

Proof. Given an S-scheme T,

$$\mathcal{F} \circ \mathcal{P}_g(T) = \mathcal{F}(T_Y).$$

The natural isomorphism η determines an isomorphism

$$\mathcal{F}(T_Y) \xrightarrow{\eta_{T_Y}} \boldsymbol{Sch}_Y(T_Y, X)$$

natural on T. Now, by Lemma 1.67, there is a bijective map natural on T

$$\boldsymbol{Sch}_Y(T_Y, X) \longrightarrow \boldsymbol{Sch}_{T_Y}(T_Y, X_T) = \mathcal{S}ect_\pi(T).$$ □

1.5.2 *Weil restrictions*

Let $Y \longrightarrow S$ be a morphism. Then, for every Y-scheme $X \xrightarrow{\pi} Y$, the *Weil restriction of* X *with respect to* $Y \longrightarrow S$ is the contravariant functor, denoted by $\mathfrak{R}_{Y/S}(X) \colon \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$, sending an S-scheme $T \longrightarrow S$ to the set

$$\mathfrak{R}_{Y/S}(X)(T) = \boldsymbol{Sch}_Y(Y_T, X).$$

This section shows Weil restrictions are equivalent to universal section families.

Theorem 1.69. *Let* $Y \longrightarrow S$ *be a morphism. Let* $X \xrightarrow{\pi} Y$ *be a* Y*-scheme. The functors* $\mathcal{S}ect_\pi$ *and* $\mathfrak{R}_{Y/S}(X)$ *are isomorphic.*

Proof. It is an immediate consequence of Lemma 1.67. □

Theorem 1.70 below can be found in [4, Theorem 4, p.194].

Theorem 1.70. *Let* S *be a ground scheme and* $\pi \colon X \longrightarrow Y$ *an* S*-morphism. If* $Y \longrightarrow S$ *is finite locally free and, for every point* s *of* S*, every finite set* P *of points on the fibre* X_s *of* $X \longrightarrow S$ *is contained in an affine open subscheme of* X*, then* $\mathcal{S}ect_{Y/S}(X)$ *is representable by a locally Noetherian quasiprojective* S*-scheme.*

Lemma 1.71 below is well-known (e.g., [45, Proposition 3.36 (b), p.109]).

Lemma 1.71. *Let* X *be quasiprojective scheme over a ring* A*. Then, every finite set* P *of points on* X *is contained in some affine open subscheme of* X*.*

1.6 FLATTENING STRATIFICATION

The flattening stratification of a morphism $X \longrightarrow S$ is a stratification $\sqcup_i S_i$ of S by locally closed subschemes (obviously unique) such that a morphism $T \longrightarrow S$ factorises through it if and only if $X_T \longrightarrow T$ is flat.

One of its main uses is in the construction of the Hilbert scheme (see Section 1.4). Indeed, fixing a polynomial $\Phi \in \mathbb{Q}[z]$, via the Castelnuovo-Mumford regularity, the functor $\mathcal{H}ilb^{\Phi}_{X/S}$ can be seen as a subfunctor of a suitable Grasmannian and the existence of the flattening stratification is asserting that such a subfunctor is representable by locally closed embeddings (see [18, §5.5.4–6, pp.128, 129]). A posteriori, it is seen that the scheme representing $\mathcal{H}ilb^{\Phi}_{X/S}$ satisfies the valuative criterion for properness, hence it is projective (see [18, §5.5.7, p.130]).

Definition 1.72. Let S be a ground scheme. Let $X \longrightarrow S$ be an S-scheme. Consider the full subcategory $\boldsymbol{F}_{X/S}$ of $\boldsymbol{Sch}_S$ whose objects are S-schemes $T \longrightarrow S$ such that $X_T \longrightarrow T$ is flat. Since the base change of a flat morphism is flat, the categories $\boldsymbol{F}_{X/S}$ and $\boldsymbol{Sch}_S$ satisfy condition $\dagger$ (see Remark 1.15.1). So, we may consider the *flattening functor for* $X \longrightarrow S$, denoted by $\flat\!\flat_{X/S} \colon \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$, defined as the contravariant characteristic functor of $\boldsymbol{F}_{X/S}$ in $\boldsymbol{Sch}_S$, that is given $(T \longrightarrow S)$ an object of $\boldsymbol{Sch}_S$,

$$\flat\!\flat_{X/S}(T \longrightarrow S) = \begin{cases} \{*\} & \text{if } X_T \longrightarrow T \text{ is flat,} \\ \emptyset & \text{otherwise.} \end{cases}$$

When X is (quasi)projective over S, given a polynomial $\Phi \in \mathbb{Q}[t]$, we may consider the subfunctor $\mathfrak{hb}^{\Phi}_{X/S}$ of the functor $\mathfrak{hb}_{X/S}$, which sends $(T \longrightarrow S) \in \boldsymbol{Sch}_S$ to

$$\mathfrak{hb}^{\Phi}_{X/S}(T \longrightarrow S) = \begin{cases} \{*\} & \text{if } X_T \longrightarrow T \text{ is flat with Hilbert poly. } \Phi \text{ on the fibres,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Similarly to the Hilbert functor, by Propositions 1.57 and 1.58, $\mathfrak{hb}_{X/S}$ is representable if and only if all the functors $\mathfrak{hb}^{\Phi}_{X/S}$ are representable, and, when they are representable, the S-scheme representing $\mathfrak{hb}_{X/S}$ is isomorphic to the disjoint union of the schemes representing $\mathfrak{hb}^{\Phi}_{X/S}$.

Theorem 1.73 below is stated as in [2, Lemma (flattening) 2.3, p.64], which is a generalisation (via the standard techniques of [22, IV3, Chapitre IV, §8, pp.5–54]) to the not necessarily Noetherian case of [25, Lemma 3.4, p.262], which in turn is the scheme theoretic version of [28, Exposé IV, Corollaire 6.11, p.104]. See [48, Lecture 8, p.55] for a detailed discussion but for Noetherian schemes.

Theorem 1.73. *Let S be a ground scheme. Let X be an S-scheme of finite presentation, locally projective over S. Let $\Phi \in \mathbb{Q}[t]$ be a polynomial. Then, the functor $\mathfrak{hb}^{\Phi}_{X/S}$ is representable and the representing scheme is a locally closed subscheme S_Φ of S. That is, a morphism $T \longrightarrow S$ factors through $S_\Phi \hookrightarrow S$ if and only if $X_T \longrightarrow T$ is flat with Hilbert polynomial Φ on the fibres.*

Remark 1.73.1. When it exists, the underlying set of S_Φ is

$$\{s \in S : \text{the Hilbert polynomial of } X_s \text{ is } \Phi\}.$$

Indeed, by definition, given a point s of S_Φ, the Hilbert polynomial of the scheme X_s is Φ. On the other hand, given a point s of S such that the Hilbert polynomial of X_s is Φ, since $X_s \longrightarrow \{s\}$ is flat, by the universal property of S_Φ, the morphism $\{s\} \hookrightarrow S$ factorises through $S_\Phi \hookrightarrow S$.

Definition 1.74. Let S be a scheme. Consider a collection of closed subscheme Z_i of S indexed by a partially ordered set I such that, set theoretically,

$$S = \bigcup_{i \in I} Z_i$$

and for all $i, j \in I$

$$Z_i \cap Z_j = \bigcup_{k \leqslant i,j} Z_k.$$

Moreover, assume that for every point s of S there is an open neighbourhood of s meeting only a finite number of schemes Z_i. Set $S_i = Z_i \setminus \left(\cup_{k \lneq i} Z_k\right)$. A such collection of locally closed subscheme S_i is said to be a *scheme theoretic stratification* (or simply a stratification) of S. It determines a monomorphism

$$\bigsqcup_{i \in I} S_i \hookrightarrow S$$

which is called the *monomorphism associated to the stratification.*

Definition 1.75. Let $X \longrightarrow S$ be a morphism. The *flattening stratification of* $X \longrightarrow S$ (which it is obviously unique, when it exists) is a stratification of S whose associated monomorphism represents the functor $\flat\flat_{X/S}$.

Theorem 1.76. *Let S be a Noetherian ground scheme. Let X be an S-scheme projective over S. Then, the flattening stratification of $X \longrightarrow S$ exists. Moreover, it can be indexed by the Hilbert polynomial $\Phi \in \mathbb{Q}[t]$ of the fibres of each strata with the partial order given by $\Phi < \Phi'$ when $\Phi(t) < \Phi'(t)$ for all $t \gg 0$. So, each stratum S_Φ is the locally closed subscheme of S representing $\flat\flat^{\Phi}_{X/S}$.*

Remark 1.76.1. By [60, Tag 01JJ], Theorem 1.76 extends straightforwardly to the case S locally Noetherian.[1]

Remark 1.76.2. By [60, Tag 05UH (4)], Theorem 1.76 extends to S any scheme and $X \longrightarrow S$ proper and flat (but in this case the strata are not necessarily indexed by the Hilbert polynomials of the fibres).

1 This remark completes the proof of [18, §5.5.6, p.129].

2 TECHNICALITIES ON BLOW UPS

"At that time, blowups were the poor man's tool to resolve singularities." This phrase of the late 21st century mathematician J.H.Φ. Leicht could become correct. In our days, however, blowups are still the main device for resolution purposes.

—Herwig Hauser
Seven short stories on blowups and resolutions

Let X be a scheme. We recall that a *locally principal subscheme* of X is a closed subscheme whose sheaf of ideals is locally generated by a single element, whereas an *effective Cartier divisor* of X is a closed subscheme whose sheaf of ideals is locally generated by a single *regular* element (see [32, Remark 6.17.1, p.145], [20, Definition 11.24, p.301], [60, Tag 01WQ] or [22, IV_4, Chapitre IV, Définition 21.1.6, p.257 and Paragraphe 21.2.12, p.262]). Given a effective Cartier divisor Z of X, we also call the closed embedding $Z \hookrightarrow X$ an effective Cartier divisor.

Let Z be a closed subscheme of X. The *blow up* of X along Z, denoted by $b\colon \mathrm{bl}(Z,X) \longrightarrow X$, is a morphism such that $b^{-1}(Z)$ is en effective Cartier divisor of $\mathrm{bl}(Z,X)$ and satisfying the following universal property: Given a morphism $f\colon T \longrightarrow X$, if $f^{-1}(Z)$ is an effective Cartier divisor of T, then it factorises through b.

The scheme Z is called the *centre* of the blow up. Its preimage $b^{-1}(Z)$ is called the *exceptional divisor* of the blow up and it is usually denoted by E.

Let $\mathcal{I}$ be the quasi-coherent $\mathcal{O}_X$-ideal cutting out Z in X. The scheme $\mathrm{bl}(Z,X)$ is

$$\mathcal{P}roj_X(\mathrm{Rees}(\mathcal{I})),$$

where $\mathrm{Rees}(\mathcal{I}) = \oplus_{n\geqslant 0}\mathcal{I}^n$ is the Rees algebra of $\mathcal{I}$ (where $\mathcal{I}^0 = \mathcal{O}_X$). The morphism $\mathrm{bl}(Z,X) \longrightarrow X$ is given by the natural morphism of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X \hookrightarrow \mathrm{Rees}(\mathcal{I}).$$

So, it is clear that the exceptional divisor E is cut out by the invertible sheaf $\mathcal{O}(1)$ of $\mathrm{bl}(Z,X)$, that is

$$E = \mathcal{P}roj_X(\bigoplus_{n>0} \mathcal{I}^n/\mathcal{I}^{n+1}).$$

For more details see any introductory book on Algebraic Geometry (e.g., [32, pp.28–31, pp.163–171], [20, pp.408–418] or [15, Chapter IV.2, pp.162–192]).

The *residual scheme* of Z in X is a construction close to the blow up. It is defined as the scheme

$$\mathrm{R}(Z,X) = \mathcal{P}roj_X(\mathrm{Sym}(\mathcal{I})),$$

where $\mathrm{Sym}(\mathcal{I}) = \oplus_{n>0} \mathrm{Sym}^n(\mathcal{I})$ is the symmetric algebra of $\mathcal{I}$. The natural morphism of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X \hookrightarrow \mathrm{Sym}(\mathcal{I})$$

gives a morphism $p\colon \mathrm{R}(Z,X) \longrightarrow X$, which is an isomorphism off Z; in fact, it is an isomorphism over every point of X at which $\mathcal{I}$ is invertible. The preimage of Z by p is cut out by the invertible sheaf $\mathcal{O}(1)$ of $\mathrm{R}(Z,X)$, that is

$$p^{-1}(Z) = \mathcal{P}roj_X(\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)).$$

There is a natural surjective morphism of graded $\mathcal{O}_X$-algebras

$$\mathrm{Sym}(\mathcal{I}) \longrightarrow \mathrm{Rees}(\mathcal{I}),$$

which gives a natural closed embedding

$$\mathrm{bl}(Z,X) \hookrightarrow \mathrm{R}(Z,X).$$

The sheaf $\mathcal{O}(1)$ of $\mathrm{R}(Z,X)$ restricts to that of $\mathrm{bl}(Z,X)$.

In this chapter we state two facts on blow ups that we will need. The first, to our knowledge is original. It states that under suitable assumptions, the blow up of a product of schemes along a locally principal subscheme preserves the product form. The second is not, it is a reformulation of when blow up commutes with arbitrary base changes.

2.1 WHEN THE CENTRE IS LOCALLY PRINCIPAL

Let S be a ground scheme and $X \longrightarrow S$ an S-scheme. We show that blowing up a locally Noetherian scheme X along a locally principal subscheme Z consists of shaving off those associated points of X lying on Z, Theorem 2.4. Given a flat S-scheme $Y \longrightarrow S$ with geometrically integral fibres, we show that there is a one-to-one correspondence, preserving specialisations, between the associated points of X and those of $X \times_S Y$, Lemma 2.6. This all yields that the blow up of $X \times_S Y$ along any locally principal subscheme is again the Cartesian product over S of Y with a closed subscheme of X, see Theorem 2.8. In particular, blowing up $X \times_S Y$ along a locally principal subscheme preserves the product form.

We use that the product form is preserved under such kind of blow ups in the construction of the blow up split section family (see Chapter 4). Indeed, such a construction consists in transforming a closed subscheme of a product of schemes into an effective Cartier divisor, similarly to blow ups but preserving the product form of the ambient scheme. So, after applying some constructions, we end up with a locally principal subscheme of a product of schemes $X \times_S Y$, where we have absolute control over the scheme Y but none for X (it is a locally closed subscheme of a suitable Hilbert scheme). Hence, in Theorem 2.8 we summarise all the required conditions on $Y \longrightarrow S$ in order to apply it as the final step of the proof of Theorem 4.3.

Let $f, g : X \longrightarrow Y$ be two morphisms and U an open subscheme of X. When U is (topologically) dense in X, the equation $f|_U = g|_V$ implies $f|_{X_{\mathrm{red}}} = g|_{X_{\mathrm{red}}}$ but not generally $f = g$.

Example 2.1. Consider X affine given by the ring $\Bbbk[x, y]/(xy, y^2)$. The scheme X consists of a line with an embedded double point at the origin p. Clearly, we may project X to a line (without the embedded point) and embed this line into X again. The composition of these two morphisms gives a morphism $f : X \longrightarrow X$ which agrees with $\mathbf{1}_X$ on the (topologically dense) open subscheme $X \setminus \{p\}$, but $f \neq \mathbf{1}_X$.

This phenomenon motivates the following definition.

Definition 2.2. Let X be a scheme. An open subscheme U of X is *scheme theoretically dense in* X (or schematically dense for short) if, for every open subscheme V of X, the schematic closure of $U \cap V$ in V is equal to V (see [60, Tag 01RB] or [22, IV_3, Chapitre IV, Définition 11.10.2, p.171]).

Remark 2.2.1. In general, there are schemes X with open subschemes U which are not schematically dense although $\overline{U} = X$ (see [60, Tag 01RC]). But, when the ambient scheme X is locally Noetherian, every open embedding is quasi-compact (see [60, Tag 01OX] or [22, I, Chapitre I, Proposition 6.6.4, p.153]) and then an open subscheme $U \hookrightarrow X$ is schematically dense if and only if $\overline{U} = X$ (see [60, Tag 01RD] or [22, IV_3, Chapitre IV, Remarque 11.10.3 (iv), p.171]).

Remark 2.2.2. When X is locally Noetherian, the schematic union of the schematic closures of all its associated points is equal to X. Hence, in this case, an open subscheme U of X is schematically dense if and only if $\mathrm{Ass}(X) \subseteq U$

Proposition 2.3. *Let* X *be a scheme and* Z *a closed subscheme of* X*. Let* $i : U \longrightarrow X$ *be the open subscheme complement of* Z *in* X *and* $b : \overline{U} \hookrightarrow X$ *its schematic closure. If* Z *is a locally principal subscheme of* X*, then the closed embedding* $b : \overline{U} \hookrightarrow X$ *is the blow up of* X *along* Z*.*

We are going to prove that if Z is an effective Cartier divisor of X, then $\overline{U} = X$ (with no assumptions on X). For the locally Noetherian case, see [22, IV_2, Chapitre IV, Corollaire 3.1.9, p.38].

Proof. Affine locally on X, we may assume Z defined by a principal ideal, say $(f) \subseteq A$ for some ring A and $f \in A$. The open embedding $U \hookrightarrow X$ is an affine morphism because affine locally it is given by $\mathrm{Spec}(A_f) \hookrightarrow \mathrm{Spec}(A)$. Therefore $U \hookrightarrow X$ is quasi-compact the sheaf $\mathcal{K} = \ker(\mathcal{O}_X \longrightarrow i_*\mathcal{O}_U)$ is quasi-coherent and, by Remark 1.46.1, it defines the closed embedding $\mathfrak{b}\colon \overline{U} \hookrightarrow X$.

Since the blow up, by its universal property, can be computed locally on X, we may assume $X \cong \mathrm{Spec}(A)$ and Z defined by $(f) \subseteq A$. Then, the open subscheme U of X is $D(f) \cong \mathrm{Spec}(A_f)$, the $\mathcal{O}_Y$-ideal $\mathcal{K}$ corresponds to the ideal $\mathfrak{a} = \ker(A \longrightarrow A_f) \subseteq A$ and the closed embedding $\mathfrak{b}$ is given by the natural homomorphism $A \longrightarrow A/\mathfrak{a}$.

When $f \in A$ is nilpotent, the subscheme U of X is the empty scheme. Moreover, for all $n \gg 0$, the n-th graded components of the Rees algebra of the ideal (f) of A are zero. Hence, the blow up of X along Z is also the empty scheme.

Assume $f \in A$ non-nilpotent. The ideal $\mathfrak{a} \subseteq A$ is $\cup_{n\in\mathbb{N}}(0 : f^n)$ (see [3, Proposition 3.11.ii, p.41]). So, the closed subscheme $\mathfrak{b}^{-1}(Z)$ of $\overline{U}$ is an effective Cartier divisor because it is defined by the class of f in $A/\mathfrak{a}$ which is a non-zerodivisor. Let $g\colon W \longrightarrow X$ be a morphism with $g^{-1}(Z)$ an effective Cartier divisor of W. Affine locally g is given by homomorphisms $\varphi\colon A \longrightarrow B$ with $\varphi(f) \in B$ a non-zerodivisor. Hence, $\mathfrak{a} \subseteq \ker(\varphi)$ and g factors through $\mathfrak{b}$. □

We have seen that blowing up along a locally principal subscheme is equivalent to taking the schematic closure of the open complement of such a locally principal subscheme. But, when the ambient scheme is locally Noetherian, there are no pathological associated points, see [60, Tag 02OI], and then, as Theorem 2.4 below shows, we can give a more explicit description of the parts that are shaved off on the blowing up procedure, which will be useful later. Namely, those associated points of the ambient scheme belonging to the centre of the blow up.

As a preparation for Theorem 2.4 below, we make the following observations. Let x be an associated point of an affine scheme X, say $X \cong \mathrm{Spec}(A)$ for some ring A, with x corresponding to a prime ideal $\mathfrak{p} \subseteq A$ (that is, $\mathfrak{p}$ is an associated prime of A, or equivalently, the maximal ideal of the stalk $\mathcal{O}_{X,x}$ is an associated prime ideal).

Given an open subscheme $U \hookrightarrow X$, since for any $y \in U$ the stalks $\mathcal{O}_{X,y}$ and $(\mathcal{O}_X|_U)_y$ are isomorphic, the sets $\mathrm{Ass}(X) \cap U$ and $\mathrm{Ass}(U)$ are equal (see [22, IV$_2$, Chapitre IV, Proposition 3.1.13, p.39]).

In contrast, given a closed subscheme $Z \hookrightarrow X$, say $Z \cong \mathrm{Spec}(A/I)$ for some ideal $I \subseteq A$, the point x belongs to Z if and only if $\sqrt{I} \subseteq \mathfrak{p}$. But now, the sets $\mathrm{Ass}(X) \cap Z$ and $\mathrm{Ass}(Z)$ are unrelated in general. For example, the case $A = \Bbbk[x,y]/(xy,y^2)$ and $I = (y) \subseteq A$ shows that $\mathrm{Ass}(X) \cap Z \nsubseteq \mathrm{Ass}(Z)$ is possible. Indeed, the scheme Z is $\mathbb{A}^1_\Bbbk$ with just one associated point, its generic point, but the point $p \in X$ corresponding to the prime ideal $(x,y) \subseteq A$ belongs to Z and it is an associated point of X. Considering the same scheme X, but now as a closed subscheme of $\mathbb{A}^2_\Bbbk$ via the natural projection $\Bbbk[x,y] \longrightarrow A$, the same point $p \in X$ shows that $\mathrm{Ass}(X) \nsubseteq \mathrm{Ass}(\mathbb{A}^2_\Bbbk) \cap X$ is also possible.

Theorem 2.4. *Let X be a locally Noetherian scheme and Z a locally principal subscheme of X. Let T_Z be the subset of X union of the underlying sets of $\overline{x}$ for all $x \in \mathrm{Ass}(X) \cap Z$. Let V be the complement of T_Z in X. Then V is an open subscheme of X and its schematic closure $\overline{V} \hookrightarrow X$ is the blow up of X along Z.*

Proof. First of all, the subset T_Z of X is closed because its intersection with every Noetherian affine open subscheme of X is a union of finitely many closed subsets (see [60, Tag 05AF] or [22, IV_2, Chapitre IV, Proposition 3.1.6, p.37]). Hence V is an open subscheme of X.

Let U be the open complement of Z and $\overline{U}$ its schematic closure. Since T_Z is a closed subset of Z, U is an open subscheme of V and of $\overline{V}$. We show that $U \hookrightarrow \overline{V}$ is schematically dense, then the claim follows from Proposition 2.3.

By definition of T_Z, $\mathrm{Ass}(X) \cap U = \mathrm{Ass}(X) \cap V$ and, by [22, IV_2, Chapitre IV, Proposition 3.1.13, p.39], $\mathrm{Ass}(\overline{V}) \subseteq \mathrm{Ass}(X) \cap V$. So, $\mathrm{Ass}(\overline{V}) \subseteq U$ and then U is a schematically dense subscheme of $\overline{V}$ by Remark 2.2.2. □

Definition 2.5. Let $\Bbbk$ be a base field. A $\Bbbk$-scheme X is called *geometrically integral* if, for every field extension $\Bbbk \hookrightarrow K$, the scheme X_K is integral. A morphism $X \longrightarrow Y$ is called *with geometrically integral fibres* if, for every point y of Y, the fibre $X_y \longrightarrow \{y\}$ is geometrically integral.

Remark 2.5.1. By [45, Chapter 3, Remark 2.9, p.90] and [62, Chapter III, Corollary 1 of Theorem 40, p.198], an integral scheme over an algebraically closed field is geometrically integral.

Lemma 2.6. *Let S be a locally Noetherian ground scheme. Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be locally Noetherian S-schemes. Let $\eta \in \mathrm{Ass}(X)$, set $s = f(\eta) \in S$ and consider the following Cartesian diagram.*

$$\begin{tikzcd}
(Y_s)_\eta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \{\eta\} \arrow[d] \\
Y_s \arrow[r] & \{s\}
\end{tikzcd}$$

Assume that g is faithfully flat and with geometrically integral fibres. Then, the scheme $(Y_s)_\eta$ is integral and its generic point is mapped to an associated point ξ_η of $X \times_S Y$ by the natural monomorphism $(Y_s)_\eta \hookrightarrow X \times_S Y$. Furthermore, the correspondence associating $\eta \in \mathrm{Ass}(X)$ with $\xi_\eta \in \mathrm{Ass}(X \times_S Y)$ is one-to-one and preserves specialisations.

Proof. The scheme Y_s is integral because we assume g with geometrically integral fibres. Denote its generic point by μ_η and denote by I_η the image of

$$\mathrm{Ass}(\mathrm{Spec}(\kappa(\eta) \otimes_{\kappa(s)} \kappa(\mu_\eta)))$$

by the natural monomorphism $\mathrm{Spec}(\kappa(\eta) \otimes_{\kappa(s)} \kappa(\mu_\eta)) \hookrightarrow X \times_S Y$. By [22, IV_2, Chapitre IV, Proposition 3.3.6, pp.44, 45],

$$\mathrm{Ass}(X \times_S Y) = \bigcup_{\eta \in \mathrm{Ass}(X)} I_\eta.$$

So now, we will show that there is a unique point in $\mathrm{Ass}(\mathrm{Spec}(\kappa(\eta) \otimes_{\kappa(s)} \kappa(\mu_\eta)))$, and then its image to $X \times_S Y$ will be the desired point ξ_η. Consider the following Cartesian diagram.

$$\begin{tikzcd} \mathrm{Spec}(\kappa(\eta) \otimes_{\kappa(s)} \kappa(\mu_\eta)) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (Y_s)_\eta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \{\eta\} \arrow[d] \\ \{\mu_\eta\} \arrow[r, hook] & Y_s \arrow[r] & \{s\} \end{tikzcd}$$

By [22, IV$_2$, Chapitre IV, Corollaire 3.3.7, p.45], there is a bijective map between $\mathrm{Ass}(\mathrm{Spec}(\kappa(\eta) \otimes_{\kappa(s)} \kappa(\mu_\eta)))$ and $\mathrm{Ass}((Y_s)_\eta)$, which has a unique point since we assume g with geometrically integral fibres.

Now, we check that the correspondence

$$\eta \in \mathrm{Ass}(X) \rightsquigarrow \xi_\eta \in \mathrm{Ass}(X \times_S Y)$$

preserves specialisations. Fix $\eta, \eta' \in \mathrm{Ass}(X)$, set $s = f(\eta)$ and $s' = f(\eta')$ and denote respectively by μ and μ' the generic points of the fibres Y_s and $Y_{s'}$. Assume that η is a specialisation of η', that is $\eta \in \overline{\eta'}$, or equivalently, affine locally the prime ideal corresponding to η' is contained in the prime ideal corresponding to η (see [20, Example 2.9, p.44]). So, it is clear that s is a specialisation of s'. Hence, we may consider the following diagram.

$$\begin{tikzcd} \{\xi_\eta\} \arrow[r] & \{\eta\} \times_s \{\mu\} \arrow[r] & \overline{\eta'} \times_{\overline{s'}} Y_{\overline{s'}} \arrow[r] & X \times_S Y \\ & & \{\eta'\} \times_{s'} \{\mu'\} \arrow[u] & \\ & & \{\xi_{\eta'}\} \arrow[u] & \end{tikzcd}$$

The schematic image of $\{\xi_{\eta'}\} \to X \times_S Y$ is the schematic image of $\overline{\eta'} \times_{\overline{s'}} Y_{\overline{s'}} \to X \times_S Y$ restricted to the schematic image of $\{\xi_{\eta'}\} \to \overline{\eta'} \times_{\overline{s'}} Y_{\overline{s'}}$ (see transitivity of schematic images, [22, I, Chapitre I, Proposition 9.5.5, p.177]). Hence, if the schematic image of $\{\xi_{\eta'}\} \to \overline{\eta'} \times_{\overline{s'}} Y_{\overline{s'}}$ is the whole ambient scheme, then the point $\{\xi_\eta\}$ is a specialisation of $\{\xi_{\eta'}\}$. It is not hard to see that the following diagram is Cartesian.

$$\begin{tikzcd} \{\eta'\} \times_{s'} Y_{s'} \arrow[r] \arrow[d] & \overline{\eta'} \times_S Y_{\overline{s'}} \arrow[r] \arrow[d] & Y_{\overline{s'}} \arrow[r] \arrow[d] & Y \arrow[d] \\ \{\eta'\} \arrow[r] & \overline{\eta'} \arrow[r] & \overline{s'} \arrow[r] & S \end{tikzcd}$$

So, since $\{\eta\} \to \overline{\eta'} \hookrightarrow \overline{\eta'}$ is a schematic image and $Y \to S$ is flat, by Lemma 1.48, the schematic image of $\{\eta'\} \times_{s'} Y_{s'} \to \overline{\eta'} \times_S Y_{\overline{s'}}$ is the whole ambient scheme. □

Lemma 2.7. *Let S be a ground scheme. Let $Y \to S$ an fpqc morphism. Let X be an S-scheme and $i\colon W \hookrightarrow X$ a closed embedding. Let $h'\colon T \to X$ be an*

S-morphism. Let $\varphi\colon T\times_S Y\longrightarrow W\times_S Y$ be a morphism such that the following diagram commutes.

$$\begin{tikzcd} T\times_S Y \arrow[r, "\varphi"] \arrow[dr, "h'_Y"'] & W\times_S Y \arrow[d, hook, "i_Y"] \\ & X\times_S Y \end{tikzcd}$$

Then, there is a unique morphism $h\colon T\longrightarrow W$ such that $\varphi = h_Y$.

Proof. Denote by $p_T\colon T\times_S Y\longrightarrow T$, $p_X\colon X\times_S Y\longrightarrow X$ and $p_W\colon W\times_S Y\longrightarrow W$ the projections. Since the following diagram commutes,

$$\begin{tikzcd} T\times_S Y \arrow[r, "i_Y\circ\varphi"] \arrow[d, "p_T"] & X\times_S Y \arrow[r, "p_X"] & X \\ T \arrow[urr, bend right, "h'"'] & & \end{tikzcd}$$

the morphism $p_X\circ i_Y\circ\varphi$ is constant along the fibres of p_T. Then, since $p_X\circ i_Y = i\circ p_W$ and i is a monomorphism, by Remark 1.50.1, the morphism $p_W\circ\varphi$ is constant along the fibres of p_T. By Proposition 1.51, there is a unique morphism $h\colon T\longrightarrow W$ such that $h\circ p_T = p_W\circ\varphi$. Consider the following diagram.

$$\begin{tikzcd} T_Y \arrow[r, "\varphi"] \arrow[d, "p_T"] & W_Y \arrow[r] \arrow[d, "p_W"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ T \arrow[r, "h"] & W \arrow[r] & S \end{tikzcd}$$

Since it commutes and both the right hand and the big squares are Cartesian, so is the left hand. Hence, $\varphi = h_Y$. □

Theorem 2.8. *Let S be a locally Noetherian ground scheme. Let $X\xrightarrow{f}S$ and $Y\xrightarrow{g}S$ be locally Noetherian S-schemes. Let Z be a locally principal subscheme of $X\times_S Y$. Assume that $Y\xrightarrow{g}S$ is flat and with geometrically integral fibres. Then, there is a closed subscheme $i\colon W\longrightarrow X$ such that the closed embedding $i_Y\colon W\times_S Y\longrightarrow X\times_S Y$ is the blow up of $X\times_S Y$ along Z.*

If furthermore $Y\longrightarrow S$ is an fpqc morphism, for every S-morphism $T\xrightarrow{h'}X$ for which the preimage of Z by $h'_Y\colon T\times_S Y\longrightarrow X\times_S Y$ is an effective Cartier divisor, there is a unique morphism $h\colon T\longrightarrow W$ such that $i\circ h' = h$. Moreover, $h_Y\colon T\times_S Y\longrightarrow W\times_S Y$ is the morphism given by the universal property of the blow up i_Y.

Proof. Let Ω denote the set of points $\xi\in \mathrm{Ass}(X\times_S Y)$ such that $\xi\in Z$. By Theorem 2.4, the blow up of $X\times_S Y$ along Z is the schematic closure of the open subscheme $U\hookrightarrow X\times_S Y$ complement of the closed subset

$$T_Z = \bigcup_{\xi\in\Omega}\overline{\xi}.$$

Let $p\colon X\times_S Y\longrightarrow X$ be the projection and denote by V the open subscheme of X complement of the closed subset

$$\bigcup_{\xi\in\Omega}\overline{p(\xi)}.$$

We claim that the schematic closure of the open embedding $V \hookrightarrow X$ is the desired closed subscheme W of X. Let us check it. Observe that g is quasi-compact because S is locally Noetherian (see [60, Tag 01OX]). Now, since g is assumed flat, by Lemma 1.48, the schematic closure of the open embedding $V \times_S Y \hookrightarrow X \times_S Y$ is $\overline{V} \times_S Y$. An associated point η of U is a point $\eta \in \mathrm{Ass}(X \times_S Y)$ such that $\eta \notin \overline{\xi}$ for all $\xi \in \Omega$. Since the one-to-one correspondence between $\mathrm{Ass}(X)$ and $\mathrm{Ass}(X \times_S Y)$ respects specialisations, this is equivalent to $p(\eta) \notin \overline{p(\xi)}$ for all $\xi \in \Omega$, which is equivalent to $\eta \in p^{-1}(V) = V \times_S Y$. Hence, $\mathrm{Ass}(U) = \mathrm{Ass}(V \times_S Y)$. Since $\xi \in p^{-1}(p(\xi))$, the scheme $V \times_S Y$ is an open subscheme of U and then the schematic closures of U and $V \times_S Y$ in $X \times_S Y$ are equal (see [22, IV_2, Chapitre IV, Proposition 3.1.13, p.39 and IV_3, Chapitre IV, Proposition 11.10.10, p.172] or [60, Tag 083P]).

Assume that $Y \longrightarrow S$ is an fpqc morphism and consider such an S-morphism $h' : T \longrightarrow X$. By the universal property of the blow up i_Y, there is a unique morphism $\varphi : T \times_S Y \longrightarrow W \times_S Y$ such that $i_Y \circ \varphi = h'_Y$. Now, the claim follows from Lemma 2.7. $\square$

Remark 2.8.1. If the assumption $Y \longrightarrow S$ with geometrically integral fibres fails, then there is a point s of S and a field extension $\kappa(s) \hookrightarrow K$ such that $(Y_s)_K$ is not integral. Setting $X = \mathrm{Spec}(K)$, the scheme $X \times_S Y$ is $(Y_s)_K$ and it has at least one locally principal subscheme Z, which is not an effective Cartier divisor. Hence, the blow up of $X \times_S Y$ along Z is not an isomorphism and, if it is not the empty scheme (otherwise Theorem 2.8 is trivial), there is no closed subscheme W of X such that $W \times_S Y \hookrightarrow X \times_S Y$ is such a blow up.

2.2 CHANGING THE BASE

This section studies when blow ups commute with arbitrary base changes. This question relies mostly on when the Rees algebra of a sheaf of ideals commutes with taking inverse images. Consider the following situation.

Situation 2.9. Let S be a ground scheme and $f : T \longrightarrow S$ a morphism. Let X be an S-scheme of finite presentation, Z a closed subscheme of X flat over S and $\mathcal{I}$ the corresponding quasi-coherent $\mathcal{O}_X$-ideal.

The inverse image of $\mathcal{I}$ by $f_X : X_T \longrightarrow X$ is the $\mathcal{O}_{X_T}$-ideal defining the closed subscheme Z_T of X_T (see Lemma 2.11). Moreover, from the assumption

♀. for every $n \gg 0$, the inverse image of $\mathcal{I}^n$ by f_X is an $\mathcal{O}_{X_T}$-ideal,

follows straightforwardly that the blow up of X along Z commutes with the base change $T \longrightarrow S$, with no additional assumptions on $T \longrightarrow S$ (see Theorem 2.12).

By Lemma 2.11, it may seem that in Situation 2.9 the assumption ♀ is already satisfied, in fact this is stated as an exercise in [61, Exercise 24.2.O, p.652], but there are counterexamples (see Remark 2.12.2).

Actually, failure of condition ☿ prevents many constructions from being functorial without assuming some conditions on the involved schemes or morphisms. In order to overcome this difficulty, we follow the idea in [38, Proposition 2.4, p.31] or [41, Proposition 3.4, p.422], which is to restrict to a case when the Rees algebra of $\mathcal{I}$ agrees with its symmetric algebra. Then, since the symmetric algebra does commute with taking inverse images, so does the Rees algebra. Nevertheless, our approach, Proposition 2.14, is slightly different from [38, Proposition 2.4, p.31]. We restrict explicitly to the case when the Rees and the symmetric algebras agree, which is called a weakly linear embedding.

Lemma 2.10. *Let $f\colon X \longrightarrow Y$ be a morphism. Let $\mathcal{J}$ be an $\mathcal{O}_Y$-ideal and n a positive integer. If, for $i = 1, n$, the $\mathcal{O}_X$-module $f^*(\mathcal{J}^i)$ is an $\mathcal{O}_X$-ideal, then $f^*(\mathcal{J}^n) = (f^*\mathcal{J})^n$.*

Proof. For $i = 1, n$, from the inclusion $\mathcal{J}^i \hookrightarrow \mathcal{O}_Y$, there is a natural morphism of $\mathcal{O}_X$-modules

$$f^*(\mathcal{J}^i) = f^{-1}\mathcal{J}^i \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow \mathcal{O}_X.$$

Clearly (and this is completely general), the ideal generated by the image of $f^*(\mathcal{J}^n) \longrightarrow \mathcal{O}_X$ is the n-th power of the ideal generated by the image of $f^*\mathcal{J} \longrightarrow \mathcal{O}_X$. But by assumption $f^*(\mathcal{J}^i) \longrightarrow \mathcal{O}_X$ are injective for $i = 1, n$, hence $f^*(\mathcal{J}^n) = (f^*\mathcal{J})^n$. □

Lemma 2.11. *In Situation 2.9, the inverse image of $\mathcal{I}$ by $f_X\colon X_T \longrightarrow X$ is a quasi-coherent $\mathcal{O}_{X_T}$-ideal and moreover, it corresponds to the closed embedding $Z_T \hookrightarrow X_T$.*

Proof. Consider the fundamental exact sequence of $\mathcal{O}_X$-modules for the closed embedding $i\colon Z \hookrightarrow X$.

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

By assumption, $i_*\mathcal{O}_Z$ is flat over S. Hence, by [20, Proposition 7.39 (1), p.194], the following sequence of $\mathcal{O}_{X_T}$-modules is exact.

$$0 \longrightarrow (f_X)^*\mathcal{I} \longrightarrow (f_X)^*\mathcal{O}_X \longrightarrow (f_X)^*i_*\mathcal{O}_Z \longrightarrow 0$$

Now, $(f_X)^*\mathcal{O}_X = \mathcal{O}_{X_T}$ (see [20, Remark 7.10, p.180]) and then $(f_X)^*\mathcal{I}$ is an $\mathcal{O}_{X_T}$-ideal. Moreover, $(f_X)^*i_*\mathcal{O}_Z = (i_T)_*\mathcal{O}_{Z_T}$ and then $(f_X)^*\mathcal{I}$ is the $\mathcal{O}_{X_T}$-ideal cutting out $Z_T \hookrightarrow X_T$. □

Theorem 2.12. *In Situation 2.9, assuming ☿, the formation of the blow up $\mathrm{bl}(Z, X)$ of X along Z commutes with base change $T \longrightarrow S$*

$$\mathrm{bl}(Z_T, X_T) = \mathrm{bl}(Z, X) \times_S T.$$

Proof. Denote by $\mathcal{I}_T$ the $\mathcal{O}_{X_T}$-ideal defining the closed subscheme Z_T of X_T.

By definition, the blow up of X along Z is the relative projective spectrum of the $\mathcal{O}_X$-algebra $\oplus_n \mathcal{J}^n$. By functoriality, there is a natural isomorphism

$$\mathcal{P}roj\left((f_X)^*\left(\bigoplus_{n\geqslant 0}\mathcal{J}^n\right)\right) \cong \mathcal{P}roj\left(\bigoplus_{n\geqslant 0}\mathcal{J}^n\right) \times_X X_T$$

(see [22, II, Chapitre II, Proposition 3.5.3, p.62]). The functor $(f_X)^*$ is left adjoint to $(f_X)_*$, hence it commutes with colimits (see [20, Proposition 7.11, p.181] and [55, Theorem 4.5.3, p.138]), so

$$(f_X)^*\left(\bigoplus_{n\geqslant 0}\mathcal{J}^n\right) = \bigoplus_{n\geqslant 0}(f_X)^*\mathcal{J}^n.$$

By Lemmas 2.10 and 2.11, for all $n \gg 0$, $(f_X)^*(\mathcal{J}^n) = ((f_X)^*\mathcal{J})^n = (\mathcal{J}_T)^n$. Finally, relative projective spectra do not depend on low degrees (see [20, §13.7, p.378] and [32, Chapter II, Exercise 2.14 (c), p.81]). □

Remark 2.12.1. In the proof of Lemma 2.11, we have showed that assumption ♀ would be satisfied if the $\mathcal{O}_X$-modules $\mathcal{O}_X/\mathcal{J}^n$ are flat over S for all $n \gg 0$. This is another approach unexplored to our knowledge.

Remark 2.12.2. Let π_1 and π_2 be two distinct planes in $\mathbb{A}^3_{\Bbbk}$ meeting along a line S. Let X be the union of π_1 and π_2 in $\mathbb{A}^3_{\Bbbk}$, so S is the singular locus of X. Consider a line Z contained in π_1 and intersecting S in exactly one point p. Consider a projection $X \longrightarrow S$ onto S, whose restriction to Z is still surjective. This example is in Situation 2.9 and, even more, $X \longrightarrow S$ is also flat. But the blow up of X along Z does *not* commute with the base change $\{p\} \longrightarrow S$.

The pathology in this example is that the stalk of the normal sheaf of Z in X at p has rank two. Moreover, even though p is a singular point of X, it is smooth on each irreducible component π_1, π_2 of X and the normal sheaf at p of $Z \cap \pi_2$ in π_1 has still rank two. But, whereas the normal sheaf of the fibre $Z_p = \{p\}$ in X_p has again rank two, the normal sheaf of Z_p in both irreducible components of X_p has just rank one.

Let us see how this affects on blowing up. The blow up of X_p along Z_p is the disjoint union of two lines, the blow up morphism is the projection onto X_p, hence its fibre at p is the disjoint union of two points.

In contrast, the blow up of X along Z is an isomorphism away from p, but its fibre at p is a whole projective line $\mathbb{P}^1_{\Bbbk}$ corresponding to the projectivisation of the normal bundle of Z at p.

Giving coordinates, it is straightforward to see that in fact the inverse image of the second power of the ideal defining Z in X is not an ideal of the corresponding ring.

Definition 2.13. Let $\sigma\colon Y \longrightarrow X$ be a closed embedding and $\mathcal{J}$ the quasi-coherent $\mathcal{O}_X$-ideal cutting out its schematic image. If the natural morphism of $\mathcal{O}_X$-modules

$$\mathrm{Sym}^n(\mathcal{J}) \longrightarrow \mathcal{J}^n$$

is an isomorphism for all $n \gg 0$, then σ is called a *weakly linear* embedding. If furthermore it is an isomorphism for all $n \geqslant 0$, then σ is called a *linear* embedding, see [37].

Proposition 2.14. *Let S be a ground scheme. Let X be an S-scheme and $\sigma\colon Z \hookrightarrow X$ a closed embedding. Then, the formation of the blow up X_σ of X along the image of σ commutes with base change $T \longrightarrow S$,*

$$(X_T)_{\sigma_T} = X_\sigma \times_S T$$

if $\sigma\colon Z \hookrightarrow X$, $\sigma_T\colon Z_T \hookrightarrow X_T$ are weakly linear and $Z \longrightarrow S$ is flat.

Proof. Denote by $\mathcal{I}$ the quasi-coherent $\mathcal{O}_X$-ideal cutting out the schematic image of σ. By Lemma 2.11, the quasi-coherent $\mathcal{O}_{X_T}$-ideal $\mathcal{I}_T$ corresponding to $\sigma_T\colon Z_T \hookrightarrow X_T$ is the inverse image of $\mathcal{I}$ by f_X. It is well-known that symmetric powers commute with taking inverse images (see [20, §11.1, pp.287, 288]), that is, since σ and σ_T are assumed weakly linear, for all $n \gg 0$,

$$(f_X)^*(\mathcal{I}^n) = (f_X)^*\left(\mathrm{Sym}^n_{\mathcal{O}_X} \mathcal{I}\right) = \mathrm{Sym}^n_{\mathcal{O}_{X_T}}\left((f_X)^*\mathcal{I}\right) = \mathrm{Sym}^n_{\mathcal{O}_{X_T}} \mathcal{I}_T = (\mathcal{I}_T)^n.$$

And the claim follows from Theorem 2.12. □

Remark 2.14.1. We have seen that when $\sigma\colon Z \hookrightarrow X$ is weakly linear, then the blow up of X along Z is the relative projective spectrum of the symmetric algebra of $\mathcal{I}$. If $Z \longrightarrow S$ is flat, from the fundamental exact sequence of the closed embedding $Z \hookrightarrow X$ follows that $\mathcal{I}$ is flat over S if and only if so is X. Hence, since the symmetric power functor preserve flat modules (see [42, Proposition 2.3, p.101]), if $\sigma\colon Z \longrightarrow X$ is weakly linear and both X, Z are flat over S, then the blow up X_σ of X along the image of σ is again flat over Z.

3 BUILDING BLOCKS

All in all it's just another brick in the wall
All in all you're just another brick in the wall

–Roger Waters
Another Brick in the Wall (Part 2)

In this chapter, we introduce two handy constructions, which will be extensively used. As far as we know, all results of this sections are new, except for Section 3.1.2, where we review two well-known cases for the representability of the functor Iso, Theorems 3.15 and 3.16.

Let S be a ground scheme. Let $p:X\longrightarrow Y$ and $f:X\longrightarrow W$ be S-morphisms. Set-theoretically, the morphism f is constant along the fibres of the morphism p if, for all point y of Y, the restriction of f to X_y, the morphism $f|_{X_y}:X_y\longrightarrow W$, is constant. Although f is not constant along the fibres of p, we may consider the (possibly empty) set Y' of points y of Y for which the morphism $f|_{X_y}$ is constant. The first construction is the f-constantify (or f-constfy for short) closed subscheme of Y, which is the scheme-theoretic construction of Y'.

Let $\pi:X\longrightarrow Y$ and $\alpha:X\longrightarrow T$ be morphisms. The second construction is the universal split section family. It parametrises sections of $\pi:X\longrightarrow Y$, but just those sections whose image is contained in some fibre of $\alpha:X\longrightarrow T$. We consider $\alpha:X\longrightarrow T$ as a morphism splitting the ambient space X by means of its fibres. So, the universal split section family is the scheme solving the parameter space problem of sections of π split by α.

3.1 CONSTFYING MORPHISMS

Let S be a ground scheme. Let $p:X\longrightarrow Y$ and $f:X\longrightarrow W$ be S-morphisms. The goal of this section is to study S-morphisms $T\longrightarrow Y$ for which the composition $X_T\longrightarrow X\xrightarrow{f} W$ is constant along the fibres of the projection $X_T\longrightarrow T$. Theorem 3.21, an immediate consequence of Theorem 3.17, shows that they form a category with a final object, which is a closed subscheme of Y. We call it the f-constantify (or f-constfy for short) closed subscheme of Y (see Definition 3.20).

We introduce the functor Iso (see Definition 3.14) mainly to study this category, but it will also have some other applications, in the study of the geometry of the blow up split section family (see Theorem 4.8) or the universal (split) section families (see Theorems 3.19 and 5.30).

The representability of the functor Iso has been studied in the literature, but explicit constructions for the representing scheme are lacking. To this end (see Theorem 3.17), we introduce the class of $\aleph_1$-projective morphisms, Definition 3.9, and its main property, Theorem 3.13. Namely, that arbitrary schematic unions of closed subschemes commute with $\aleph_1$-projective pullbacks.

3.1.1 *Base change of schematic unions*

In this section, we introduce $\aleph_1$-projective morphisms to show that arbitrary schematic unions of closed subschemes commute with $\aleph_1$-projective pullbacks.

Definition 3.1. Let R be a ring. An R-module M is *Mittag-Leffler* if the natural homomorphism

$$\rho\colon M\otimes_R \prod_{i\in I} Q_i \longrightarrow \prod_{i\in I} M\otimes_R Q_i$$

is injective for every family of R-modules $(Q_i \mid i\in I)$.

Example 3.2. For example, finitely presented modules are Mittag-Leffler. A finitely generated module is Mittag-Leffler if and only if it is finitely presented. Projective modules are also Mittag-Leffler, in particular, so are free modules.

On the other side, a typical example of a non-Mittag-Leffler module is $\mathbb{Q}$ as a $\mathbb{Z}$-module. Indeed, consider the family of $\mathbb{Z}$-modules $Q_n = \mathbb{Z}/n\mathbb{Z}$. So, $\prod_n \mathbb{Q}\otimes_{\mathbb{Z}} Q_n = 0$. But, $\mathbb{Q}$ is a subring of $\mathbb{Q}\otimes_{\mathbb{Z}}\prod_n Q_n$. Since $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module, applying $\mathbb{Q}\otimes_{\mathbb{Z}}\_$ to the injective homomorphism $\mathbb{Z}\hookrightarrow\prod_n Q_n$, we get $\mathbb{Q}\hookrightarrow\mathbb{Q}\otimes_{\mathbb{Z}}\prod_n Q_n$.

For more examples see [60, Tag 059Q].

We are interested in Mittag-Leffler modules which moreover are flat. In [33], there is a complete characterisation of such modules as $\aleph_1$-projective modules, which motivates Definition 3.7 below (see [33, Corollary 2.7, p.3443 and Corollary 2.10, p.3444]). We review the main definitions and results for the convenience of the reader.

Definition 3.3. Let R be a ring, and M a R-module. Let κ be a regular uncountable cardinal. A direct system $\mathcal{C}$ of submodules of M is said to be a *κ-dense system in M* if

(1) $\mathcal{C}$ is closed under unions of well-ordered ascending chains of length smaller than κ, and

(2) every subset of M of cardinality smaller than κ is contained in an element of $\mathcal{C}$.

Theorem 3.4 (see [33, Corollary 2.7, p.3443]). *Let R be a ring and M a module. Then, M is Mittag-Leffler if and only if there is an $\aleph_1$-dense system in M consisting of countably generated pure projective modules.*

Definition 3.5. Let R be a ring, and let κ be a regular uncountable cardinal. An R-module M is said to be κ-projective if there is a κ-dense system $\mathcal{C}$ consisting of projective modules generated by less than κ elements.

Theorem 3.6 (see [33, Corollary 2.10, p.3444]). *Let R be a ring. An R-module M is flat and Mittag-Leffler if and only if it is $\aleph_1$-projective.*

Definition 3.7. We say that an homomorphism $\varphi\colon A \longrightarrow B$ is *$\aleph_1$-projective* if B is an $\aleph_1$-projective A-module via φ.

Lemma 3.8. *Let $A \longrightarrow B$ be an $\aleph_1$-projective homomorphism. Then, for every family of ideals $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ of A,*

$$B \cdot \bigcap_{\lambda\in\Lambda} \mathfrak{a}_\lambda = \bigcap_{\lambda\in\Lambda} B \cdot \mathfrak{a}_\lambda.$$

Proof. Since B is a flat A-module, the following sequence is exact.

$$0 \hookrightarrow B \otimes_A \textstyle\bigcap_{\lambda\in\Lambda} \mathfrak{a}_\lambda \hookrightarrow B \otimes_A A \xrightarrow{\alpha} B \otimes_A \displaystyle\prod_{\lambda\in\Lambda} A/\mathfrak{a}_\lambda$$

So, $B \cdot \cap_\lambda \mathfrak{a}_\lambda = \ker(\alpha)$. Now, since B is a Mittag-Leffler A-module, the natural homomorphism

$$\rho\colon B \otimes_A \prod_{\lambda\in\Lambda} A/\mathfrak{a}_\lambda \longrightarrow \prod_{\lambda\in\Lambda} B \otimes_A A/\mathfrak{a}_\lambda$$

is injective. Hence, $\ker(\alpha) = \ker(\rho \circ \alpha) = \cap_\lambda B \cdot \mathfrak{a}_\lambda$. □

Definition 3.9. Let $f\colon X \longrightarrow Y$ be morphism. An *affine cover of* f is a couple $(\mathcal{U}, \mathcal{V})$ where $\mathcal{U} = \{U_i\}_i$ is an affine open cover of Y and $\mathcal{V} = \{V_{i,j}\}_{i,j}$ is a collection of affine open covers $\{V_{i,j}\}_j$ of $f^{-1}(U_i)$ for ever i. An *$\aleph_1$-projective cover of* f is an affine cover $(\mathcal{U}, \mathcal{V})$ of f such that for every i, j the homomorphism corresponding to $V_{i,j} \longrightarrow U_i$ is $\aleph_1$-projective. We say that $f\colon X \longrightarrow Y$ is *$\aleph_1$-projective*, if it admits an $\aleph_1$-projective covering.

Remark 3.9.1. The existence of an $\aleph_1$-projective cover of f does *not* imply that every affine cover of f is $\aleph_1$-projective. The existence of an $\aleph_1$-projective cover can be computed locally on the target, since the union of $\aleph_1$-projective covers is again an $\aleph_1$-projective cover. The property of being $\aleph_1$-projective is stable under pullbacks by affine morphisms. But in general, since preimages do not preserve affineness, it is not clear whether $\aleph_1$-projectivity of morphisms is preserved under arbitrary pullback or base changes (even though flat and Mittag-Leffler modules ascend along arbitrary ring maps).

Example 3.10. Let $\Bbbk$ be a field. Let X, Y be $\Bbbk$-schemes. Then, the projection $X \times_\Bbbk Y \longrightarrow X$ is $\aleph_1$-projective. Fix affine covers $\mathcal{U} = \{U_i\}$, $\{V_j\}$ of X, Y respectively. Then, the set $\mathcal{V} = \{U_i \times V_j\}$ is an affine cover of $X \times Y$ and the couple $(\mathcal{U}, \mathcal{V})$ is an $\aleph_1$-projective covering of $X \times Y \longrightarrow X$. Let us check it.

For every i, j, the projection $U_i \times V_j \longrightarrow U_i$ corresponds to the natural homomorphism $A \longrightarrow A \otimes_\Bbbk B$ for some $\Bbbk$-algebras A, B. So, $A \otimes_\Bbbk B$ is a free A-module and free modules are flat (well-known) and Mittag-Leffler (see [60, Tag 059Q]).

Example 3.11. In fact, in Example 3.10 we have seen that every morphism $f: Z \longrightarrow S$ which, affine locally can be given by homomorphisms $A \longrightarrow B$ with B a free A-module, is $\aleph_1$-projective. In particular, if f is an affine morphism such that the $\mathcal{O}_S$-module $f_*\mathcal{O}_Z$ is locally free.

Notation 3.12. Let X be a scheme. Consider a family of closed subschemes Y_l of X cut off by a family of quasi-coherent $\mathcal{O}_X$-ideals $\{\mathcal{I}_l\}_l$. We denote by $\Sigma_l Y_l$ its schematic union, that is, the closed subscheme of X corresponding to the quasi-coherent $\mathcal{O}_X$-ideal $\bigcap_l \mathcal{I}_l$.

Theorem 3.13 below is the main property for which we introduce $\aleph_1$-projective morphisms. It asserts that arbitrary schematic unions of closed subscheme commute with $\aleph_1$-projective pullbacks.

Theorem 3.13. *Let $X \longrightarrow Y$ be an $\aleph_1$-projective morphism. Then, for every family $\{Y_l\}_l$ of closed subschemes of Y, the closed subschemes $X_{\Sigma_l Y_l}$ and $\Sigma_l X_{Y_l}$ of X are equal.*

Proof. Fix an $\aleph_1$-projective covering $(\{U_i\}, \{V_{i,j}\})$ of $X \longrightarrow Y$. We check that for every i, j the closed subschemes $(X_{\Sigma_l Y_l}) \cap V_{i,j}$ and $(\Sigma_l X_{Y_l}) \cap V_{i,j}$ of $V_{i,j}$ are equal.

Fix i, j and denote respectively by A and B the rings of functions of U_i and $V_{i,j}$. Every closed subscheme $Y_l \cap U_i$ of U_i is given by an ideal $\mathfrak{a}_l$ of A. The closed subschemes $(X_{\Sigma_l Y_l}) \cap V_{i,j}$ and $(\Sigma_l X_{Y_l}) \cap V_{i,j}$ of $V_{i,j}$ are given respectively by the ideals $\cap_l B \cdot \mathfrak{a}_l$ and $B \cdot \cap_l \mathfrak{a}_l$. But since B is an $\aleph_1$-projective A-module by assumption, by Lemma 3.8, such ideals are equal. □

3.1.2 *The Iso functor*

Let S be a ground scheme. Let $X \longrightarrow Y$ be an S-morphism. In this section, we study morphisms $T \longrightarrow S$ for which the base change $X_T \longrightarrow Y_T$ is an isomorphism. We review that sending T to the set of such morphisms defines a functor, Definition 3.14, and the main cases where the representability of such a functor has been studied, Theorems 3.15 and 3.16.

We use the introduced class of $\aleph_1$-projective morphisms to give an explicit description of the representing scheme, Theorem 3.17. To finish, we show (via Hilbert schemes) that when X is flat over S, the functor Iso is isomorphic to the functor $\flat^{\Phi}$ for a suitable polynomial Φ.

Definition 3.14. Let $p: X \longrightarrow Y$ and $Z \longrightarrow X$ be morphisms. Consider the full subcategory of $\boldsymbol{Sch}_Y$ consisting of Y-schemes $T \longrightarrow Y$ such that $Z_T \longrightarrow X_T$ is an isomorphism. Since isomorphisms are stable by base change, such categories satisfy condition ⸸ (see Remark 1.15.1) and we define Iso_p^Z: $\boldsymbol{Sch}_Y \longrightarrow \boldsymbol{Set}$ as their contravariant characteristic functor. That is, it sends an Y-scheme $T \longrightarrow Y$ to

$$\mathit{Iso}_p^Z(T) = \begin{cases} \{*\} & \text{if } Z_T \longrightarrow X_T \text{ is an isomorphism,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Let us illustrate the definition of the Iso^Z_p functor in the easiest case in which it is representable by an affine algebraic scheme. Fix a base field $\Bbbk$, consider the rings $A = \Bbbk[A_1, \ldots, A_n]$ and $R = A \otimes_{\Bbbk} \Bbbk[x_1, \ldots, x_m]$, and assume $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(A)$ and $p\colon X \longrightarrow Y$ given by the natural homomorphism $A \hookrightarrow R$. A closed subscheme Z of X is given by polynomial equations $\{p_i = 0\}_i$ with $p_i \in R$. So, when we wonder for which $(A_1, \ldots, A_n) \in \Bbbk^n$ the equations $\{p_i = 0\}$ are satisfied *for all* $(x_1, \ldots, x_m) \in \Bbbk^m$, we write down the polynomials p_i as elements of $A[x_1, \ldots, x_m]$. That is, $p_i = \sum_j a^i_j \mathbf{x}^j$, where $j = (j_1, \ldots, j_m)$, $a^i_j \in A$ and $\mathbf{x}^j = x_1^{j_1} \cdots x_m^{j_m}$. Then, for the $(A_1, \ldots, A_n) \in \Bbbk^n$ satisfying $\{a^i_j = 0\}_{i,j}$ the polynomials p_i are identically zero and the equations $\{p_i = 0\}_i$ are satisfied for all $(x_1, \ldots, x_m) \in \Bbbk^m$.

A bit more formally, consider the closed subscheme W of Y given by the equations $\{a^i_j = 0\}_{i,j}$. We are just saying that the base change of the closed embedding $Z \hookrightarrow X$ by $W \hookrightarrow Y$ is an isomorphism and that W is the "biggest" closed subscheme of Y with this property. In this case, W represents Iso^Z_f.

Remark 3.14.1. Notice that a closed embedding has a section if and only if it is an isomorphism. Hence, if $Z \longrightarrow X$ is a closed embedding, then $\mathit{Iso}^Z_p = \mathit{Sect}_{Z \longrightarrow X} = \mathfrak{R}_{X/Y}(Z)$ (see Sections 1.5 and 1.5.2).

Remark 3.14.2. If the functor Iso^Z_p is representable by an open or closed subscheme Y' of Y, the underlying set of Y' is

$$\omega = \{y \in Y \text{ such that } Z_y \longrightarrow X_y \text{ is an isomorphism}\}.$$

Indeed, if a point y of Y belongs to Y', then $Z_y \longrightarrow X_y$ is the base change of (the isomorphism) $Z_{Y'} \longrightarrow X_{Y'}$ by $y \longrightarrow Y'$, hence $y \in \omega$. If $y \in \omega$, then, by the universal property of the closed embedding $Y' \hookrightarrow Y$, the morphism $\{y\} \longrightarrow Y$ factorises through $Y' \hookrightarrow Y$. Hence, y belongs to Y'.

There are two main different cases when the representability of the functor Iso^Z_p has been studied. We state them for the convenience of the reader.

The following can be found in [60, Tag 07AI].

Theorem 3.15. *Let $p\colon X \longrightarrow Y$ be a morphism and $Z \hookrightarrow X$ a closed embedding. If p is of finite presentation, flat, and pure, then Iso^Z_p is representable and the representing scheme Y' is a closed subscheme of Y. Moreover, if $Z \longrightarrow Y$ is of finite presentation, then so is $Y' \hookrightarrow Y$.*

Theorem 3.16 below, by Proposition 1.15, is equivalent to [18, Theorem 5.22 (b), p.132].

Theorem 3.16. *Let $p\colon X \longrightarrow Y$ and $Z \longrightarrow X$ be morphisms. If Y is Noetherian, $Z \longrightarrow X$ is projective and Z, X are proper and flat over Y, then Iso^Z_p is representable in the category of locally Noetherian Y-schemes and the representing scheme Y' is an open subscheme of Y.*

Remark 3.16.1. By [60, Tag 01JJ], Theorem 3.16 extends straightforwardly to the case Y locally Noetherian.[1]

[1] This remark completes the proof of [18, Theorem 5.23, p.133].

Remark 3.16.2. Notice that a proper morphism onto a locally Noetherian scheme is of finite presentation (trivially) and pure (see [60, Tag 05K3]). Hence, if Y locally Noetherian and furthermore $Z \longrightarrow X$ is a closed embedding, by Theorem 3.15, the scheme Y' representing $\mathcal{I}so_p^Z$ is a union of connected components of Y.

The representability of the functor $\mathcal{S}ect_\pi$ claimed in Proposition 1.62 can now be easily established as a corollary:

Proof of Proposition 1.62. We show that $\mathcal{S}ect_\pi$ is a subfunctor of $\mathcal{H}ilb_{X/S}$ representable by open embeddings, then the claim follows by Theorem 1.59 and Lemma 1.17

Given an S-scheme T, denote by $p_T : Y_T \longrightarrow T$ the base change of the structure morphism, which is flat and proper because so is $Y \longrightarrow S$. Given $(\sigma : Y_T \longrightarrow X_T) \in \mathcal{S}ect_\pi T$, the composition $p_T \circ \pi_T \circ \sigma$ is again p_T, hence it is flat and proper and, since σ is a closed embedding (see Proposition 1.49), σ itself is an element of $\mathcal{H}ilb_{X/S}T$. So, we have defined an injective map $\mathcal{S}ect_\pi T \hookrightarrow \mathcal{H}ilb_{X/S}T$, which is clearly natural on T, that is we have showed that $\mathcal{S}ect_\pi$ is a subfunctor of $\mathcal{H}ilb_{X/S}$.

Now, we show that $\mathcal{S}ect_\pi \hookrightarrow \mathcal{H}ilb_{X/S}$ is representable by open embeddings. An element $Z \in \mathcal{H}ilb_{X/S}T$ corresponds to a closed embedding $i : Z \hookrightarrow X_T$ such that the composition $Z \hookrightarrow X_T \longrightarrow T$ is proper and flat. By Remark 1.76.1, there is an open subscheme U_Z of T representing the functor $\mathcal{I}so^Z_{Y_T \longrightarrow T}$ and by Proposition 1.15 it satisfies the following universal property: For every locally Noetherian T-scheme T', the base change $i' : Z_{T'} \hookrightarrow X_{T'}$ of i by $T' \longrightarrow T$ composed with $\pi_{T'} : X_{T'} \longrightarrow Y_{T'}$ is an isomorphism (that is, i' can be identified with a section of $\pi_{T'}$) if and only if $T' \longrightarrow T$ factors through $U_Z \hookrightarrow T$. □

Theorem 3.17. *Let $p : X \longrightarrow Y$ be a morphism and Z a closed subscheme of X. Let Ω denote the set of closed subschemes W of Y such that $Z_W \hookrightarrow X_W$ is an isomorphism and denote by Σ_Ω the closed subscheme $\Sigma_{W \in \Omega} W$ of Y. If p is $\aleph_1$-projective, then the scheme Σ_Ω represents the functor $\mathcal{I}so_p^Z$.*

By Proposition 1.15, a closed subscheme Y' of Y represents the functor $\mathcal{I}so_p^Z$ if and only if a morphism $T \longrightarrow Y$ factorises through $Y' \hookrightarrow Y$ whenever the closed embedding $Z_T \hookrightarrow X_T$ is an isomorphism.

Proof of Theorem 3.17. For every $W \in \Omega$, the isomorphism $Z_W \hookrightarrow X_W$ is an X-morphism, hence the closed embeddings $Z_W \hookrightarrow X$ and $X_W \hookrightarrow X$ correspond to the same closed subscheme of X. Therefore, the schemes $\Sigma_{W \in \Omega} Z_W$ and $\Sigma_{W \in \Omega} X_W$ are the same subscheme of X and, by Theorem 3.13, the closed embedding $Z_{(\Sigma_\Omega)} \hookrightarrow X_{(\Sigma_\Omega)}$ is an isomorphism, in fact an X-isomorphism. So, if a morphism $T \longrightarrow Y$ factorises through Σ_Ω, the closed embedding $Z_T \hookrightarrow X_T$ is an isomorphism.

Now, given a morphism $T \longrightarrow Y$ such that the closed embedding $Z_T \hookrightarrow X_T$ is an isomorphism, by Lemma 1.47, the schematic image $\overline{T}$ of $T \longrightarrow Y$ is a closed subscheme of Y belonging to Ω. Hence, the closed embedding $\overline{T} \hookrightarrow Y$

factorises (obvious uniquely) through $\Sigma_\Omega \hookrightarrow Y$ and then, by composition, so does $T \longrightarrow Y$. □

Theorem 3.18. *Let S be a locally Noetherian ground scheme. Let $p: X \longrightarrow Y$ be an S-morphism, with X quasiprojective over S, and let Z be a closed subscheme of X. If p is projective and flat and Y Noetherian and connected, then the Hilbert polynomial of the fibres of p is constant, say equal to Φ. If furthermore Y is separated over S, then the functors Iso^Z_p and $\flat\!\flat^{\Phi}_{Z/Y}$ are isomorphic. Moreover, they are representable and the representing scheme is a closed subscheme Y' of Y whose underlying set is*

$$\{y \in Y \text{ such that } Z_y \hookrightarrow X_y \text{ is an isomorphism}\}.$$

Proof. Since p is projective (in particular, proper) it is pure and of finite presentation (see Remark 3.16.2). Hence, by Theorem 3.15, the functor Iso^Z_p is representable by a closed subscheme Y' of Y. By [22, III$_2$, Chapitre III, Proposition 7.9.11, Corollaire 7.9.13, p.79] the Hilbert polynomial of the fibres of p is constant. Therefore, there is a natural transformation from Iso^Z_p to $\flat\!\flat^{\Phi}_{Z/Y}$. Instead of giving its inverse explicitly, we will relegate its existence to powerful well-known constructions. Indeed, we will show that the scheme Y' is the fibre product of Y with suitable Hilbert schemes and that such fibre product satisfies the universal property determining $\flat\!\flat^{\Phi}_{Z/Y}$ (see Proposition 1.15). Notice that this is also a proof by its own that Iso^Z_p is representable in this case.

Consider the Hilbert functor $\mathcal{H}_{X/S}$ of $X \longrightarrow S$ and fix a morphism $(T \longrightarrow Y) \in h_Y(T)$. By Lemma 1.31, the following diagram is Cartesian.

$$\begin{tikzcd} X \times_Y T \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \times_S T \arrow[d] \\ Y \arrow[r, hook, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd} \tag{3.1.1}$$

Then, since $Y \longrightarrow S$ is separated, $X \times_Y T \hookrightarrow X \times_S T$ is a closed embedding. Since $p: X \longrightarrow Y$ is flat and proper, so is its pullback $X \times_Y T \longrightarrow T$ by $T \longrightarrow Y$, which is equal to the composition of $X \times_Y T \hookrightarrow X \times_S T$ with the projection $X \times_S T \longrightarrow T$. Hence, $X \times_Y T$ belongs to $\mathcal{H}_{X/S}(T)$ and now it is straightforward to see that sending $(T \longrightarrow Y) \in h_Y(T)$ to $X \times_Y T \in \mathcal{H}_{X/S}(T)$ defines a natural transformation $\eta: h_Y \longrightarrow \mathcal{H}_{X/S}$.

Consider the Hilbert functor $\mathcal{H}_{Z/S}$ of $Z \longrightarrow \mathbb{k}$. We claim that the functor Iso^Z_p is isomorphic to the fibre product $h_Y \times_{\mathcal{H}_{X/S}} \mathcal{H}_{Z/S}$, where $h_Y \longrightarrow \mathcal{H}_{X/S}$ is the natural transformation η just defined, and $\mathcal{H}_{Z/S} \hookrightarrow \mathcal{H}_{X/S}$ is the natural closed embedding. Then, it is representable because $\mathcal{H}_{X/S}$ and $\mathcal{H}_{Z/S}$ are representable (see [18, Theorem 5.14, p.127]) and the representing scheme is a closed subscheme of Y because $\mathcal{H}_{Z/S}$ is a closed subfunctor of $\mathcal{H}_{X/S}$ (see [18, Lemma 5.17 (ii), p.127]).

Let us check that Iso^Z_p is isomorphic to $h_Y \times_{\mathcal{H}_{X/S}} \mathcal{H}_{Z/S}$. Fix a morphism $(T \longrightarrow Y) \in \mathit{Iso}^Z_p(T)$. So, the base change $Z \times_Y T \longrightarrow X \times_Y T$ of $Z \hookrightarrow X$

by $T \longrightarrow Y$ is an isomorphism. Now, it is almost immediate to see that the couple $((T \longrightarrow Y), Z \times_Y T)$ is an element of $h_Y(T) \times_{\mathcal{H}_{X/S}(T)} \mathcal{H}_{Z/S}(T)$ and such assignment defines a natural transformation τ from $h_Y \times_{\mathcal{H}_{X/S}} \mathcal{H}_{Z/S}$ to $\mathcal{I}so^Z_p$.

Fix a couple $((T \longrightarrow Y), \Lambda)$ belonging to $h_Y(T) \times_{\mathcal{H}_{X/S}(T)} \mathcal{H}_{Z/S}(T)$, that is, $X \times_Y T$ and Λ are isomorphic as X-schemes. Then, since Λ is a closed subscheme of $Z \times_S T$, via the projection $Z \times_S T \longrightarrow Z$ we get an X-morphism from $X \times_Y T$ to Z. So, by Lemma 1.32, the closed embedding $Z \times_Y T \hookrightarrow X \times_Y T$ is an isomorphism, $(T \longrightarrow Y) \in \mathcal{I}so^Z_p(T)$ and such assignment defines a natural transformation σ from $\mathcal{I}so^Z_p$ to $h_Y \times_{\mathcal{H}_{X/S}} \mathcal{H}_{Z/S}$.

Clearly by construction, the natural transformations τ and σ are inverse to each other.

Let H_X, H_Z denote respectively the schemes representing the functors $\mathcal{H}_{X/S}$ and $\mathcal{H}_{Z/S}$. By weak Yoneda's lemma, Lemma 1.7, there are morphisms $Y \longrightarrow H_X$, $H_Z \hookrightarrow H_X$ and $Y' \longrightarrow H_Z$ corresponding to the Equation (3.1.1). So, the following diagram is Cartesian.

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & H_Z \arrow[d] \\
Y \arrow[r] & H_X
\end{tikzcd}$$

Notice that the morphisms $Y \longrightarrow H_X$, $Y' \longrightarrow H_Z$ factorise through the stratum corresponding to the Hilbert polynomial Φ.

Let $T \longrightarrow Y$ be a morphism such that $Z_T \longrightarrow T$ is flat and the Hilbert polynomial of the fibres is Φ. Again Z_T is a closed subscheme of $Z \times_S T$ and $Z_T \longrightarrow T$ is proper because $Z \longrightarrow Y$ is so, hence $Z_T \in \mathcal{H}_{Z/S}T$. Then, by the universal property of H_Z, there is a unique morphism $T \longrightarrow H_Z$ such that Z_T is the pullback by it of the universal family of H_Z. Finally, by the universal property of the pullback, there is a unique morphism $T \longrightarrow Y'$ such that the corresponding diagram commutes.

Now, for every point y of Y', the closed embedding $Z_y \hookrightarrow X_y$ is an isomorphism. Moreover, by Remark 1.73.1, the underlying set of Y' is

$$\{y \in Y \text{ such that the Hilbert polynomial of } Z_y \text{ is } \Phi\}.$$

Hence, if $y \notin Y'$, then the Hilbert polynomial of Z_y is different from the one of X_y and $Z_y \hookrightarrow X_y$ is not an isomorphism. □

Let S be a ground scheme. As a first application, we will show how to retrieve the universal section family of an S-morphism $\pi\colon X \longrightarrow Y$ restricted to a closed subscheme Z of X from the universal section family of π.

Theorem 3.19. *Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-morphism and $i\colon Z \hookrightarrow X$ a monomorphism. Assume that the universal section family $(\mathfrak{X}, \psi)$ of π exists. Then, the functors $\mathcal{S}ect_{\pi|_Z}$ and $\mathcal{I}so^{\psi^{-1}(Z_{\mathfrak{X}})}_{Y_{\mathfrak{X}} \longrightarrow \mathfrak{X}}$ are isomorphic. In particular, they are equivalently representable and, when they are representable, they are represented by the same scheme.*

Proof. Let T be an S-scheme. Given a T-family of sections $\sigma\colon Y_T \longrightarrow Z_T$ of $\pi|_Z$, by Lemma 1.32, the pullback φ of $Z_T \hookrightarrow X_T$ by $i_T \circ \sigma$ is an isomorphism. Since $i_T \circ \sigma$ is a T-family of sections of π, by the universal property of $(\mathfrak{X}, \psi)$, there is a unique morphism $f_\sigma\colon T \longrightarrow \mathfrak{X}$ such that $\psi_T = i_T \circ \sigma$. Now, it is straightforward to check that the base change of $\psi^{-1}(Z_{\mathfrak{X}}) \hookrightarrow Y_{\mathfrak{X}}$ by f_σ is the isomorphism φ. Hence,

$$\mathit{Iso}^{\psi^{-1}(Z_{\mathfrak{X}})}_{Y_{\mathfrak{X}} \longrightarrow \mathfrak{X}}(T) = \{*\}.$$

On the other hand, given a morphism $T \longrightarrow \mathfrak{X}$ such that the base change of $\psi^{-1}(Z_{\mathfrak{X}}) \hookrightarrow Y_{\mathfrak{X}}$ by it is an isomorphism, the inverse of this isomorphism gives a T-family of sections $Y_T \hookrightarrow Z_T$. ☐

3.1.3 *The constfy closed subscheme*

Finally, everything is ready to quickly present the constfy closed subscheme.

Definition 3.20. Let S be a ground scheme. Let $p\colon X \longrightarrow Y$ and $f\colon X \longrightarrow W$ be S-morphisms. Let Y' be a closed subscheme of Y. We call Y' a f-*constfy* closed subscheme of Y, if the morphism $f|_{X_{Y'}}\colon X_{Y'} \longrightarrow W$ is constant along the fibres of the projection $X_{Y'} \longrightarrow Y'$ and it satisfies the following universal property: A morphism $T \longrightarrow Y$ factorises through $Y' \hookrightarrow Y$ if and only if $(X_T \longrightarrow X)^*(f)$ is constant along the fibres of the projection $X_T \longrightarrow T$.

If a f-constfy closed subscheme exists, by abstract nonsense it is uniquely determined up to a unique isomorphism and therefore it is unique.

Remark 3.20.1. Consider the following Cartesian.

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[d, hook, "\Delta_{W/S}"] \\
X \times_Y X \arrow[r, "f \times_Y f"] & W \times_S W
\end{tikzcd}$$

Consider Z as a Y-scheme. For every morphism $T \longrightarrow Y$, the following diagram is Cartesian (see Lemma 1.30).

$$\begin{tikzcd}
Z_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[d, "\Delta_{W/S}"] \\
X_T \times_T X_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \times_Y X \arrow[r, "f \times_S f"] \arrow[d] & W \times_S W \\
T \arrow[r] & Y &
\end{tikzcd}$$

Theorem 3.21. *Let S be a ground scheme. Let $p\colon X \longrightarrow Y$ and $f\colon X \longrightarrow W$ be S-morphisms. Consider the following Cartesian diagram.*

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[d, hook, "\Delta_{W/S}"] \\
X \times_Y X \arrow[r, "f \times_Y f"] & W \times_S W
\end{tikzcd}$$

Set $g\colon X\times_Y X\longrightarrow Y$. *If* W *is separated over* S *and* p *is flat and proper, then the* f*-constfy closed subscheme of* Y *exists and it is the scheme representing the functor* Iso_g^Z.

Proof. Since W is separated, $Z\hookrightarrow X\times_Y X$ is a closed embedding and, since p is flat and proper, so is $g\colon X\times_Y X\longrightarrow Y$. Hence, by Theorem 3.15, the functor Iso_g^Z is represented by a closed subscheme Y' of Y.

By Remark 3.20.1, it is clear that $Z_{Y'}\hookrightarrow(X_{Y'}\times_{Y'}X_{Y'})$ is an isomorphism, and then $f|_{X_{Y'}}$ is constant along the fibres of the projection $X_{Y'}\longrightarrow Y'$. Furthermore, by the universal property of Y' and again by Remark 3.20.1, it is clear that a morphism $T\longrightarrow Y$ factorises through $Y'\hookrightarrow Y$ if and only if $Z_T\hookrightarrow(X_T\times_T X_T)$ is an isomorphism, that is, if and only if $(X_T\longrightarrow X)^*(f)$ is constant along the fibres of the projection $X_T\longrightarrow T$. □

Remark 3.21.1. Let S be a ground scheme. Let $p\colon X\longrightarrow Y$ and $f\colon X\longrightarrow W$ be S-morphisms. Let $Z\hookrightarrow X$ be a closed subscheme of X. In this situation, we may compose the constructions of the f-constfy closed subscheme of Y and the closed subscheme of Y representing the functor Iso_p^Z. Assuming existence, it is straightforward to see that both possible ways of composing such constructions give the same closed subscheme of Y.

3.2 SPLIT SECTIONS IN FAMILY

In this section we study a variant of the universal section family. Consider the following situation.

Situation 3.22. Let S be a Noetherian ground scheme. Let X, Y and T be S-schemes with $Y \longrightarrow S$ an fpqc morphism. Let $\pi: X \longrightarrow Y$ and $\alpha: X \longrightarrow T$ be S-morphisms. Denote by $\pi': X \longrightarrow Y_T$ the product of π and α.[1]

$$\begin{tikzcd}
X \arrow[dr, "\pi'"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\pi"'] & & \\
& Y_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T \arrow[d] \\
& Y \arrow[r, "\textbf{fpqc}"] & S
\end{tikzcd}$$

Our goal is parametrise sections of $\pi: X \longrightarrow Y$, but just those sections whose image is contained in some fibre of $\alpha: X \longrightarrow T$. We consider $\alpha: X \longrightarrow T$ as a morphism splitting the ambient space X by means of its fibres. So, the universal split section family is the scheme solving the parameter space problem of sections of π whose image is contained in some fibre of α.

Definition 3.23. Consider Situation 3.22. Let T' be an S-scheme and $q: X_{T'} \longrightarrow X$ the projection. We say that a T'-family of sections σ over π is T-*split*, if the morphism $Y_{T'} \xrightarrow{\sigma} X_{T'} \xrightarrow{q} X \xrightarrow{\alpha} T$ is constant along the fibres of the projection $p: Y_{T'} \longrightarrow T'$. In this case, we also call σ a T'-*family of* T-*split sections over* π.

$$\begin{tikzcd}
X_{T'} \arrow[r, "\pi_{T'}"] \arrow[d, "q"] & Y_{T'} \arrow[r, "p"] \arrow[d] \arrow[l, dashed, bend left, "\sigma"] & T' \arrow[d] \\
X \arrow[r, "\pi"] \arrow[dr, "\alpha"] & Y \arrow[r] & S \\
& T \arrow[ur] &
\end{tikzcd}$$

Let $\mathfrak{Y}$ be an S-scheme and ψ a $\mathfrak{Y}$-family of T-split sections over π. We call the couple $(\mathfrak{Y}, \psi)$ a *universal* T-*split section family of* π (or T-Ussf for short) if it satisfies the following universal property: For every S-scheme T' and every T'-family of T-split sections σ over π, there is a unique S-morphism $h: T' \longrightarrow \mathfrak{Y}$ such that $h_X \circ \sigma = \psi \circ h_Y$, or equivalently, such that the following diagram is Cartesian.

$$\begin{tikzcd}
Y_{T'} \arrow[r, hook, "\sigma"] \arrow[d, "h_Y"'] \arrow[dr, phantom, "\ulcorner", very near start] & X_{T'} \arrow[r, "\pi_{T'}"] \arrow[d, "h_X"] \arrow[dr, phantom, "\ulcorner", very near start] & Y_{T'} \arrow[d, "h_Y"] \\
Y_{\mathfrak{Y}} \arrow[r, hook, "\psi"] & X_{\mathfrak{Y}} \arrow[r, "\pi_{\mathfrak{Y}}"] & Y_{\mathfrak{Y}}
\end{tikzcd}$$

1 The morphism π' plays no role until Proposition 3.27.

Remark 3.23.1. Let T'' be an S-scheme and σ' a T''-family of T-split sections over π. As it happens for families of sections over π, the collection of families of T-split sections over π form a category $\boldsymbol{S}_\pi^T$ where an arrow from σ to σ', denoted by $(T', \sigma) \longrightarrow (T'', \sigma')$, is a morphism $h: T'' \longrightarrow T'$ such that $h_X \circ \sigma' = h_Y \circ \sigma$, or equivalently, such that σ' is the base change of σ by h. This is the same notion of an arrow in the category of families of sections over π (see Remark 1.60.1 and the proof of Proposition 1.61), that is, the category $\boldsymbol{S}_\pi^T$ is a full subcategory of $\boldsymbol{S}_\pi$.

If a T-Ussf exists, by abstract nonsense it is uniquely determined up to a unique isomorphism.

Remark 3.23.2. Since $Y \longrightarrow S$ is an fpqc morphism, so is $Y_{T'} \longrightarrow T'$ and, by Proposition 1.51, a T'-family of sections σ over π is T-split if and only if there is a (unique) S-morphism $f: T' \longrightarrow T$ such that $\alpha \circ q \circ \sigma = f \circ p$, or in other words, if and only if T' is a T-scheme and p is a T-morphism, where $Y_{T'}$ is a T-scheme via $\alpha \circ q \circ \sigma$.

Let $q': X_{T''} \longrightarrow T''$ and $p': Y_{T''} \longrightarrow T''$ be the projections and $f': T'' \longrightarrow T$ the unique morphism such that $\alpha \circ q' \circ \sigma' = f' \circ p'$. Then, if a morphism $h: T'' \longrightarrow T'$ determines an arrow $(T', \sigma) \longrightarrow (T'', \sigma')$ of families of T-split sections, then, by the uniqueness of f', the following diagram commutes.

$$\begin{tikzcd} T'' \arrow[rr, "h"] \arrow[dr, "f'"'] & & T' \arrow[dl, "f"] \\ & T & \end{tikzcd}$$

Definition 3.24. In Situation 3.22, we define $\mathcal{S}ect_\pi^T: \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$, the contravariant functor corresponding to the parameter space problem of T-split sections over π, as follows. For every S-scheme T', set

$$\mathcal{S}ect_\pi^T T' = \{\sigma \in \mathcal{S}ect_\pi(T') \text{ such that } \sigma \text{ is T-split}\} \subseteq \boldsymbol{Sch}_{Y_T}(Y_T, X_T),$$

and for every S-morphism $f: T'' \longrightarrow T'$, the map $\mathcal{S}ect_\pi^T f: \mathcal{S}ect_\pi^T T' \longrightarrow \mathcal{S}ect_\pi^T T''$ sends a T'-family of T-split sections $\sigma: Y_{T'} \longrightarrow X_{T'}$ over π to its base change $\sigma_{T''}: Y_{T''} \longrightarrow X_{T''}$ by f, which clearly is a T''-family of T-split sections over π.

Lemma 3.25. *Consider Situation 3.22. Assume that the Usf $(\mathfrak{X}, \psi')$ of π exists (so, $\mathfrak{X}$ is an S-scheme and ψ' is a section of $\pi_{\mathfrak{X}}: X_{\mathfrak{X}} \longrightarrow Y_{\mathfrak{X}}$ satisfying the corresponding universal property). Denote by $q: X_{\mathfrak{X}} \longrightarrow X$ the projection. If the $(\alpha \circ q \circ \psi')$-constfy closed subscheme $\mathfrak{Y}$ of $\mathfrak{X}$ also exists, then the T-Ussf of π is the couple $(\mathfrak{Y}, \psi)$ where $\psi = \psi'|_{Y_{\mathfrak{Y}}}$.*

Proof. By construction the couple $(\mathfrak{Y}, \psi)$ is a $\mathfrak{Y}$-family of T-split sections over π. So, we just need to check that it satisfies the required universal property.

Let T' be an S-scheme and σ a T'-family of T-split sections over π. Since it is a T'-family of sections over π, by the universal property of $(\mathfrak{X}, \psi')$, there is a unique morphism $h' \colon T' \longrightarrow \mathfrak{X}$ such that the following diagram is Cartesian.

$$
\begin{tikzcd}
Y_{T'} \arrow[r, hook, "\sigma"] \arrow[d, "h'_Y"] \arrow[dr, phantom, "\ulcorner", very near start] & X_{T'} \arrow[r] \arrow[d, "h'_X"] \arrow[dr, phantom, "\ulcorner", very near start] & T' \arrow[d, "h'"] \\
Y_{\mathfrak{X}} \arrow[r, hook, "\psi'"] & X_{\mathfrak{X}} \arrow[r] \arrow[d, "\alpha\circ q"] & \mathfrak{X} \\
& T &
\end{tikzcd}
$$

By assumption σ is T-split, that is $\alpha \circ q' \circ \sigma \colon Y_{T'} \longrightarrow T$ (where $q' \colon X_{T'} \longrightarrow X$ is the projection) is constant along the fibres of the projection $Y_{T'}$, but $q' = q \circ h'_X$ so that the restriction $(\alpha \circ q \circ \psi')|_{Y_{T'}}$ is constant along the fibres of $Y_{T'} \longrightarrow T'$. Hence, by the universal property of $\mathfrak{Y}$, there is a unique morphism $h \colon T' \longrightarrow \mathfrak{Y}$ whose composition with the closed embedding $\mathfrak{Y} \hookrightarrow \mathfrak{X}$ is h'. □

Theorem 3.26. *Consider Situation 3.22 with S locally Noetherian. If T is separated, X is at most a countable disjoint union of quasiprojective schemes over S and Y proper and flat over S, then the T-Ussf of π exists and its underlying scheme is locally Noetherian and at most a countable disjoint union of quasiprojective schemes.*

Proof. By Proposition 1.63, the Usf $(\mathfrak{X}, \psi')$ of π exists and $\mathfrak{X}$ is locally Noetherian and at most a countable disjoint union of quasiprojective schemes. Denote by $q \colon X_{\mathfrak{X}} \longrightarrow X$ the projection. Now, since $Y \longrightarrow S$ is of finite presentation, flat and proper (in particular, it is also pure see Remark 3.16.2), so is $Y_{\mathfrak{X}} \longrightarrow \mathfrak{X}$. Hence, by Theorem 3.21, the $(\alpha \circ q \circ \psi')$-constfy closed subscheme of $\mathfrak{X}$ exists and the claim follows from Lemma 3.25. □

Proposition 3.27. *Consider Situation 3.22. Let $f \colon T' \longrightarrow T$ be an S-morphism. Let $\sigma \colon Y_{T'} \longrightarrow X_{T'}$ be a T'-family of sections over π. Denote by $p \colon Y_{T'} \longrightarrow T'$ and $q \colon X_{T'} \longrightarrow X$ the projections. Then, σ is T-split with $\alpha \circ q \circ \sigma = f \circ p$ if and only if $f_Y = \pi' \circ (q \circ \sigma)$, that is if and only if the following diagram commutes.*

$$
\begin{tikzcd}
X \arrow[drrrr, bend left, "\alpha"] \arrow[drr, "\pi'"] \arrow[ddrr, bend right, "\pi"'] & & & & \\
& & Y_T \arrow[rr] \arrow[d] & & T \arrow[d] \\
& & Y \arrow[rr] & & S \\
& Y_{T'} \arrow[rr, "p"'] \arrow[ur] \arrow[uur, dashed, "f_Y"] \arrow[uuul, bend left, "q\circ\sigma"] & & T' \arrow[ur] \arrow[uur, dashed, "f"] &
\end{tikzcd}
$$

Proof. Since morphisms to Y_T are determined by their composition with the projections $Y_T \longrightarrow T$ and $Y_T \longrightarrow Y$, from σ being T-split with $\alpha \circ q \circ \sigma = f \circ p$ follows straightforwardly that $f_Y = \pi' \circ (q \circ \sigma)$.

If $f_Y = \pi' \circ (q \circ \sigma)$, then composing with the projection $Y_T \longrightarrow Y$ follows that $f \circ p = \alpha \circ q \circ \sigma$ and, by Remark 3.23.2, σ is T-split. □

Remark 3.27.1. Notice that σ is equal to the product of $(q \circ \sigma)$ and p and π is the composition of π' with the projection $Y_T \longrightarrow Y$. Hence, the T'-family of sections σ over π is completely determined by $(q \circ \sigma)$ and π'. Moreover, by Proposition 3.27, the morphism f allows us to check at once whether σ is T-split or not. So, for convenience in the forthcoming sections, whenever $f_Y = \pi' \circ (q \circ \sigma)$, we will refer also to a couple $((q \circ \sigma), f)$ as a T'-family of T-split sections over π'. Moreover, when the T-Ussf $(\mathfrak{Y}, \psi)$ of π exists, there is the corresponding morphism $h \colon \mathfrak{Y} \longrightarrow T$ (and the projection $q' \colon X_{\mathfrak{Y}} \longrightarrow X$), in this case we also call the triplet $(\mathfrak{Y}, q' \circ \psi, h)$ the T-Ussf of π'.

4

THE BLOW UP SPLIT SECTION FAMILY

Consider the following situation.

Situation 4.1. Let S be a locally Noetherian ground scheme. Let X, Y be locally Noetherian S-schemes with $Y \longrightarrow S$ an fpqc morphism. Consider the scheme $X_Y = X \times_S Y$ and denote by $\pi\colon X_Y \longrightarrow Y$ and $\alpha\colon X_Y \longrightarrow X$ the projections. Let Z be a closed subscheme of X_Y.

$$\begin{tikzcd}
Z \arrow[r, hook, "\textbf{cl.emb.}"] & X_Y \arrow[r, "\alpha"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\
& Y \arrow[r, "\textbf{fpqc}"] & S
\end{tikzcd}$$

The main result of this chapter is Theorem 4.3, which asserts the existence of the blow up split section family of the projection $X_Y \longrightarrow Y$ along Z (see Definition 4.2) under suitable assumptions. The blow up split section family is a generalisation of blow ups, as such Theorem 4.8 is the corresponding generalisation of the well-known fact that a blow up is an isomorphism away from its centre.

Definition 4.2. Consider Situation 4.1. Let $\mathfrak{B}$ be an S-scheme and $b\colon \mathfrak{B} \longrightarrow X$ an S-morphism.

$$\begin{tikzcd}
(b_Y)^{-1}(Z) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{B}_Y \arrow[r] \arrow[d, "b_Y"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{B} \arrow[d] \\
Z \arrow[r, hook] & X_Y \arrow[r, "\alpha"] & X
\end{tikzcd}$$

We call the couple $(\mathfrak{B}, b)$ a *blow up split section family of* π *along* Z (or blow up §family for short) if $(b_Y)^{-1}(Z) \hookrightarrow \mathfrak{B}_Y$ is an effective Cartier divisor and it satisfies the following universal property: For every S-morphism $g\colon T \longrightarrow X$ for which $(g_Y)^{-1}(Z) \hookrightarrow T_Y$ is an effective Cartier divisor, there is a unique morphism $h\colon T \longrightarrow \mathfrak{B}$ such that $b \circ h = g$. Analogously to classic blow ups, we call Z the *centre* of the blow up §family and $b^{-1}(X_Y)$ the *exceptional divisor* in $\mathfrak{B}_Y$.

For examples see Section 4.1, in particular Example 4.12, which is computed by hand.

If a blow up §family exists, by abstract nonsense it is uniquely determined up to a unique isomorphism.

Theorem 4.3. *Consider Situation 4.1. If moreover X_Y is at most a countable disjoint union of quasiprojective schemes over S, $X \longrightarrow S$ is separated and $Y \longrightarrow S$ is proper and with geometrically integral fibres, then the blow up §family of π along Z exists.*

Proof. Consider the blow up $\mathrm{bl} : \mathrm{bl}(Z, X_Y) \longrightarrow X_Y$ of X_Y along Z. The scheme $\mathrm{bl}(Z, X_Y)$ is again at most a countable disjoint union of quasiprojective schemes over S. Hence, by Theorem 3.26, the X-Ussf $(\mathfrak{Y}, \psi, \nu)$ of $\mathrm{bl}(Z, X_Y) \longrightarrow Y$ exists (so, $\psi : Y_{\mathfrak{Y}} \longrightarrow \mathrm{bl}(Z, X_Y)$, $\nu : \mathfrak{Y} \longrightarrow X$ are morphisms such that $\mathrm{bl} \circ \psi = \nu_Y$ and the scheme $\mathfrak{Y}$ is locally Noetherian and at most a countable disjoint union of quasiprojective schemes).

Finally, since the preimage of Z by ν_Y is the preimage by ψ of the exceptional divisor in $\mathrm{bl}(Z, X_Y)$, it is a locally principal subscheme of $Y_{\mathfrak{Y}}$ and, by Theorem 2.8, there is a closed subscheme $\mathfrak{B}$ of $\mathfrak{Y}$ such that the closed embedding $\mathfrak{B}_Y \hookrightarrow Y_{\mathfrak{Y}}$ is the blow up of $Y_{\mathfrak{Y}}$ along $(\nu_Y)^{-1}(Z)$. Denote by $\mathfrak{b} : \mathfrak{B} \longrightarrow X$ the restriction of ν to $\mathfrak{B}$.

Now, it is straightforward to check that the couple $(\mathfrak{B}, \mathfrak{b})$ is the blow up §family of π along Z. It follows by applying iteratively the universal properties of the objects used to construct $\mathfrak{B}$ and, at the last step, Theorem 2.8. □

The blow up §family can be defined, and a suitable version of Theorem 4.3 proved, when in Situation 4.1 the square is not necessarily Cartesian. Nevertheless, we restrict to this case for the sake of simplicity and because this is the unique case we will use. Furthermore, Theorem 4.8 below uses that such square is Cartesian.

Consider Situation 4.1 assuming X connected and Y integral and projective over S. In this situation, we may state Theorem 4.8 below, the generalisation to blow up §families of the well-known fact that a blow up is an isomorphism away from its centre. We need to introduce some notation and, for the convenience of the reader, we also introduce Lemma 4.4 below, which extracts the parts of [18, Lemma 9.3.4, p.258] and [60, Tag 062Y] we are interested in.

Lemma 4.4. *Let X be a ground scheme. Let X' be a flat X-scheme locally of finite presentation (e.g., if X is locally Noetherian). Let Z be a closed subscheme of X' and z a point of Z with image x in X.*

(a) If $Z \longrightarrow X$ is locally of finite presentation and flat, and the fibre $Z_x \hookrightarrow X'_x$ is an effective Cartier divisor, then Z is an effective Cartier divisor of X' in an open neighbourhood of z. In particular, if all the fibres $Z_x \hookrightarrow X'_x$ are effective Cartier divisors, then Z is an effective Cartier divisor of X'.

(b) If Z is a locally principal subscheme of X' and all the fibres $Z_x \hookrightarrow X'_x$ are effective Cartier divisors, then Z is an effective Cartier divisor of X' and flat over X.

Notation 4.5. Consider Situation 4.1 assuming X connected and Y projective over S (so $X_Y \longrightarrow X$ is also projective). The flattening stratification of the morphism $Z \longrightarrow X$ is a finite stratification

$$X = \sqcup_{\Phi \in \mathbb{Q}[t]} X_\Phi$$

by locally closed subschemes such that for every Φ, the pullback of $Z \longrightarrow X$ by $X_\Phi \hookrightarrow X$ is flat and the Hilbert polynomial of the fibres is constant equal to Φ, and moreover, a morphism $T \longrightarrow X$ factorises through $\sqcup_\Phi X_\Phi \hookrightarrow X$ if and only if the projection $Z_T \longrightarrow T$ is flat (see Section 1.6).

Since $X_Y \longrightarrow X$ is flat and projective and X is connected, the Hilbert polynomial of its fibres is constant, say equal to $\Phi_0 \in \mathbb{Q}[t]$. By Theorem 3.15 and Remark 3.16.2, the functor $\mathit{Iso}^Z_{X_Y \longrightarrow X}$ is representable by a closed subscheme X_0 of X. Observe that, by Remark 3.14.2, the underlying sets of X_{Φ_0} and X_0 are equal. In fact, under slightly stronger assumptions they are isomorphic, see Theorem 3.18.

By Lemma 4.4 (a), for every Φ, the points $x \in X_\Phi$ for which $Z_x \hookrightarrow (X_Y)_x = Y_x$ is an effective Cartier divisor form a (possibly empty) open subscheme of X_Φ; we denote it by U_Φ.

We present a simple example showing that, with no extra assumptions, even if all the fibres $Z_x \hookrightarrow (X_Y)_x$ are effective Cartier divisors, Z is not necessarily an effective Cartier divisor of X_Y. Consider X an affine line, $\mathbb{A}^1_\Bbbk$ for some base field $\Bbbk$. Consider Y the projective line $\mathbb{P}^1_\Bbbk$ over $\Bbbk$. For Z any closed point z of X_Y, every fibre $Z_x \hookrightarrow (X_Y)_x$ is either $\emptyset \hookrightarrow \mathbb{P}^1_{\kappa(z)}$ or $\{z\} \hookrightarrow \mathbb{P}^1_\Bbbk$, which both are effective Cartier divisors.

Denote by x_0 the image of z by $Z \longrightarrow X$ and set $X_1 = X \setminus \{x_0\}$ and $X_2 = \{x_0\}$. Regarding Notation 4.5, observe that $X = X_1 \sqcup X_2$ is the flattening stratification of $Z \longrightarrow X$ because, for $i = 1, 2$, the scheme Z_{X_i} is flat over X_i, and moreover it is an effective Cartier divisor in $(X_Y)_{X_i}$.

Remark 4.5.1. By [26, Théorème 2.1 (i), p.231], if furthermore Y is smooth over S, the open subscheme U_Φ of X_Φ is also a closed subset. Hence, U_Φ is either the empty scheme or a union of connected component of X_Φ.

Definition 4.6. Consider Situation 4.1 assuming X connected and Y projective over S. We call a point x of X *type I* if x belongs to some U_Φ and *type II* otherwise. In particular, if Y is smooth over S, by Remark 4.5.1, each connected component of each stratum X_Φ is filled up with either type I or type II points. In this case, we also call respectively each connected component of each stratum *type I* or *type II*.

Definition 4.7. Consider Situation 4.1 assuming X connected and Y projective over S. Consider also Notation 4.5. We call the scheme X_0 the *core* of the blow up §family of π along Z.

Theorem 4.8. *Consider Situation 4.1 assuming X connected and Y integral and projective over S. Consider also Notation 4.5. Assume that the blow up §family $(\mathfrak{B}, b)$ of π along Z exists. Then, the open subscheme $\mathfrak{B} \setminus b^{-1}(X_0)$ of $\mathfrak{B}$ is isomorphic to $\sqcup_\Phi U_\Phi$.*

Proof. Denote by E the exceptional divisor in $\mathfrak{B}_Y$, that is $E = (b_X)^{-1}(Z)$. The closed subscheme $b^{-1}(X_0)$ of $\mathfrak{B}$ represents the functor $\mathit{Iso}^E_{\mathfrak{B}_Y \longrightarrow \mathfrak{B}}$, hence $\mathfrak{B} \setminus b^{-1}(X_0)$ is the set of points $b \in \mathfrak{B}$ for which $E_b \hookrightarrow Y_b$ is not an isomorphism. Then, since $E \hookrightarrow \mathfrak{B}_Y$ is an effective Cartier divisor and Y is integral, $\mathfrak{B} \setminus b^{-1}(X_0)$ is the open subset corresponding to the set of points $b \in \mathfrak{B}$ for which $E_b \hookrightarrow Y_b$ is an effective Cartier divisor. Then, by Lemma 4.4 (b), $E \cap (\mathfrak{B} \setminus b^{-1}(X_0)) \longrightarrow \mathfrak{B} \setminus b^{-1}(X_0)$ is flat and then, by the universal property of the flattening stratification, there is a unique morphism $\mathfrak{B} \setminus b^{-1}(X_0) \longrightarrow \sqcup_\Phi X_\Phi$ (whose image clearly is contained in $\sqcup_\Phi U_\Phi$) such that the corresponding diagram commutes. Hence, it factorises through $\sqcup_\Phi U_\Phi \hookrightarrow \sqcup_\Phi X_\Phi$ via a unique morphism $\xi\colon (\mathfrak{B} \setminus b^{-1}(X_0)) \longrightarrow \sqcup_\Phi U_\Phi$.

Now, by construction and by Lemma 4.4 (a), $Z_{\sqcup_\Phi U_\Phi} \hookrightarrow X_{\sqcup_\Phi U_\Phi}$ is an effective Cartier divisor, hence, by the universal property of $(\mathfrak{B}, b)$, there is a unique morphism $\sqcup_\Phi U_\Phi \longrightarrow \mathfrak{B}$ (whose image is contained in $\mathcal{B} \setminus b^{-1}(X_0)$ because U_{Φ_0} is empty) such that the corresponding diagram commutes. So finally, $\sqcup_\Phi U_\Phi \longrightarrow \mathfrak{B}$ factorises through a unique morphism $\varepsilon\colon \sqcup_\Phi U_\Phi \longrightarrow (\mathfrak{B} \setminus b^{-1}(X_0))$.

Now, it is straightforward to check that ξ and ε are mutually inverse. □

Corollary 4.8.1. *Consider Situation 4.1 assuming X connected and Y integral and projective over S. If there is no point x of X such that the fibre $Z_x \hookrightarrow Y_x$ is an isomorphism, then the blow up §family of π along Z exists and it is the natural morphism $\sqcup_\Phi U_\varphi \hookrightarrow X$.*

Proof. In this case the core of the blow up §family is empty. □

Corollary 4.8.2. *For every irreducible component B of $\mathfrak{B}$, if $b(B) \not\subseteq X_0$, then B is birational to an irreducible component of the closure of a stratum X_Φ. More explicitly, $b|_B : B \longrightarrow \overline{X_\Phi}$ decomposes as $B \overset{i}{\hookrightarrow} B' \xrightarrow{b'} \overline{X_\Phi}$ where i is an open embedding and b' is a blow up morphism whose centre fails to be Cartier only on the core X_0. In particular, if the closure of the stratum X_Φ, for some Hilbert polynomial Φ, does not intersect X_0, then $b|_B$ is an open embedding.*

Now, we present an example showing that in Remark 4.5.1 the assumption X smooth over S is required. It is based in Hartshorne's example of a flat family of rational normal curves with a singular fibre [32, Example 9.8.4, p.259]. We focus on the relevant affine chart. Set $A = \Bbbk[a]$, $R = \Bbbk[x, y, z]$ and $B = (A \otimes_\Bbbk R)/I$ where

$$I = \Big(a^2(x+1) - z^2,\ ax(x+1) - yz,\ xz - ay,\ y^2 - x^2(x+1)\Big).$$

Consider $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and $X \longrightarrow S$ given by the natural homomorphism $A \longrightarrow B$. Since A is a pid and $A \longrightarrow B$ has no torsion, B is a flat A-module and $X \longrightarrow S$ is flat. It is not hard to check that X is singular at the origin (a, x, y, z), hence $X \longrightarrow S$ is not smooth.

Observe that when $a \neq 0$,

$$\frac{1}{a}\Big(z \cdot (ay - xz) + x \cdot (a^2(x+1) - z^2)\Big) = ax(x+1) - yz$$
$$\frac{1}{a^2}\Big((ay + xz) \cdot (ay - xz) - x^2 \cdot (a^2(x+1) - z^2)\Big) = y^2 - x^2(x+1).$$

Hence, when $a \neq 0$, the fibre X_a of the family $X \longrightarrow S$ corresponds to the ring R/I_a, where

$$I_a = \Big(ay - xz,\ a^2(x+1) - z^2\Big),$$

which is a rational normal curve. Instead, the fibre X_0 at $a = 0$ corresponds to the ring R/I_0 where

$$I_0 = \left(z^2, yz, xz, y^2 - x^2(x+1)\right),$$

which is nodal curve with a non-reduced structure at the origin. Now, consider $C = (R \otimes_{\Bbbk} A)/J$ where

$$J = \left(x,\ y,\ a - z\right).$$

Observe that $I \subseteq J$. Indeed, except $a^2(x+1) - z^2$, all the other generator of I belong to the ideal (x, y) and

$$a^2 \cdot (x) + (a+z) \cdot (a-z) = a^2(x+1) - z^2.$$

Consider $Z = \mathrm{Spec}(C)$, which corresponds to a line $C \cong \Bbbk[a, z]/(a-z)$, and consider $Z \longrightarrow X$ given by the natural homomorphism $B \longrightarrow C$. So, the composition $Z \longrightarrow S$ is flat and smooth. Now, when $a \neq 0$, the fibre Z_a of $Z \longrightarrow S$ is a point corresponding to the ring R/J_a where formally $J_a = J$ but now a is an element of $\Bbbk$, that is

$$R/J_a \cong \Bbbk[z]/(a-z).$$

Observe that when $a \neq 0$,

$$\begin{aligned} \frac{1}{a^2}\Big(-(a+z)\cdot(a-z) + (a^2(x+1) - z^2)\Big) &= x \\ \frac{1}{a}\Big(z \cdot (x) - (xz - ay)\Big) &= y. \end{aligned}$$

hence $J_a/I_a = (z-a) \subseteq R/I_a$ where $z - a$ is a non-zerodivisor of R/I_a, and then $Z_a \hookrightarrow X_a$ is an effective Cartier divisor. Instead, when $a = 0$, the ideal J_0 is (x, y, z), and the ideal $J_0/I_0 = (x, y, z) \subseteq R/I_0$ is generated by zerodivisors and it is not even principal. Hence, $Z_0 \hookrightarrow X_0$ is not an effective Cartier divisor.

Observe that the ring of functions of Z_0 is reduced, since Z_0 is a flat limit of a family of single simple points, but it is also possible to get a non reduced scheme as a flat limit of effective Cartier divisor. Indeed, consider the ideal $(x, y, a^2 - z^2) \subseteq R \otimes_{\Bbbk} A$, which also contains I. It corresponds to a family in X flat and non-smooth over S, for which the fibre at $a \neq 0$ is a union of two simple points (which form an effective Cartier divisor of X_a), but the fibre at $a = 0$ is a double point (which is not an effective Cartier divisor of X_0).

4.1 EXAMPLES

In this section, we recover two classic constructions, the classic blow up (see Proposition 4.10) and an example of a small resolution, both as particular cases of the blow up §family.

We also present an example showing that the blow up §family may also behave quite different from such classic constructions, namely, the dimension of the ambient scheme may decrease.

Example 4.9 (The classic blow up)**.** The following proposition shows the classic blow up as a particular case of the blow up §family.

Proposition 4.10. *Consider Situation 4.1. Assume that there is a closed subscheme W of X such that $Z = W_Y$. Let $b\colon \mathfrak{B} \longrightarrow X$ be the blow up of X along W. Then, the couple $(\mathfrak{B}, b)$ is the blow up §family of $\pi\colon X_Y \longrightarrow Y$ along W_Y. In particular, when $\beta = \mathbf{1}_S$, the blow up §family agrees with the classic blow up.*

Proof. It follows straightforwardly from the fact that $Y \longrightarrow S$ is flat and blow ups commute with flat base changes (see [60, Tag 0805]). ☐

Example 4.11 (The dimension may decrease)**.** We show an example of the blow up §family where an irreducible ambient space breaks down into two irreducible components and the dimension of one of them decreases by one.

Consider $\mathcal{S} = \mathbb{P}^1_{u,v} \times \mathbb{P}^2_{x,y,z}$ and $Z \subseteq \mathcal{S}$ the graph of $[u:v] \in \mathbb{P}^1 \longrightarrow [u:v:0] \in \mathbb{P}^2$, that is $Z = V_+(z, vx - uy)$.

By Corollary 4.8.1, the blow up §family of the projection $\mathcal{S} \longrightarrow \mathbb{P}^1$ along Z is the stratification of $\mathbb{P}^2$ by the standard affine chart $\mathbb{P}^2 \setminus V_+(z)$ and $V_+(z)$.

Example 4.12 (Small resolution)**.** We present an example where the blow up §family along a natural centre becomes a small resolution. It indicates the possibility that the blow up §family would offer a procedure to systematise small resolutions.

Let $\Bbbk$ be a field and consider the variety $\mathbb{A}^4_{\Bbbk}$ parametrising matrices

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

and the closed subvariety $D \subseteq \mathbb{A}^4$ where the rank of M is not maximal, or equivalently where the determinant of M is zero. Consider the variety $\mathcal{S} = \mathbb{P}^1_{u,v} \times D$ and its incidence subvariety

$$Z = \{([\lambda], M) \in \mathcal{S} : M\lambda^t = 0\}.$$

It is a classic result that the projection $\mathcal{S} \longrightarrow D$ restricted to Z is a small resolution of D. It turns out that the blow up §family of the projection $\mathcal{S} \longrightarrow \mathbb{P}^1$ along Z is isomorphic to Z and then again an small resolution of D.

Observe that, by Theorem 4.8, the variety $D \setminus \{0\}$ is an open subvariety of such a blow up §family. But we do not retrieve the whole ambient variety from this result. Instead, we replicate the construction of the blow up §family in Theorem 4.3.

First, let us construct the following quasiprojective varieties V_n. Let S denote the standard graded polynomial ring $\Bbbk[u,v]$ and S_n its degree n part. So, we define $V_n \subseteq \mathbb{P}(S_n \times S_n \times S_n)$ as the quasiprojective variety corresponding to triplets of forms with no common roots.

The blow up $\tilde{\mathcal{S}}$ of $\mathcal{S}$ along Z may be given globally by the equations $xa - zb$ and $ya - wb$ in $\mathcal{S} \times \mathbb{P}^1_{a,b}$.

Now, we describe the closed subvariety of the universal section family of $\tilde{\mathcal{S}} \longrightarrow \mathbb{P}^1$ corresponding to "constfy" by $\tilde{\mathcal{S}} \longrightarrow D$. Namely, it is the disjoint union X for all integers n of the closed subvarieties X_n of $D \times V_n$ determined by the equations on the coefficients given by the identities of polynomials,

$$
\begin{aligned}
xA - zB &\equiv 0 \\
yA - wB &\equiv 0
\end{aligned}
$$

where $[A : B] \in V_n$ (see example after Definition 3.14). The resulting morphism $b' \colon X \longrightarrow D$ is for each component X_n the composition of the closed embedding $X_n \hookrightarrow D \times V_n$ and the projection $D \times V_n \longrightarrow D$.

It is straightforward to see that given $((x,y,z,w),[A : B]) \in X_n$ either the forms A, B are constants or $(x,y,z,w) = 0$. That is, for all $n \geqslant 1$, $X_n = \{0\} \times V_n$ and

$$
X = X_0 \coprod \Big(\coprod_{n \geqslant 1} \{0\} \times V_n \Big)
$$

where $X_0 \cong Z$. So, the pullback $(\mathbf{1}_{\mathbb{P}^1} \times b')^{-1}(Z)$ is an effective Cartier divisor in $\mathbb{P}^1 \times X_0$ and the whole $\mathbb{P}^1 \times X_n$ for all $n \geqslant 1$. Hence the blow up of $\mathbb{P}^1 \times X$ along the locally principal $(\mathbf{1}_{\mathbb{P}^1} \times b')^{-1}(Z)$ is $\mathbb{P}^1 \times X_0$, and then the blow up §family of $\mathcal{S} \longrightarrow \mathbb{P}^1$ along Z is $b'|_{X_0} \colon X_0 \longrightarrow D$.

5 CLUSTERS IN FAMILY

Let S be a ground scheme and π an S-morphism. In this chapter, we introduce the main notions of this memoir, length-r *clusters* over π (see Definition 5.7), *families of clusters* (see Definitions 5.8 and 5.9) and their parameter spaces Cl^r (see Definition 5.16). Our construction of such parameter spaces requires that certain blow ups commute with arbitrary base changes. To this end, we need to impose some regularity conditions on π, which leads us to the notion of *steady S-family* (see Definition 5.5).[1]

We define the schemes Cl^r via universal properties, so our first result, Theorem 5.19, is its existence under finiteness assumptions on π. In Section 5.4 we show that the blow up §family is the iterative step relating the scheme Cl^{r+1} with the scheme Cl^r, Theorem 5.37. More precisely, the blow up §family $\mathfrak{B}$ of $Cl^r_{Cl^{r-1}} Cl^r$ along a suitable closed subscheme is a closed subscheme of Cl^{r+1} (see Corollary 5.35.1), which parametrises pairs of clusters of π and their flat limits. There also is a closed subscheme $(Cl^{r+1})_E$ of Cl^{r+1} parametrising those clusters of π whose $r+1$-th section is infinitely near to the r-th (see Theorem 5.30). And Theorem 5.37 shows that $(Cl^{r+1})_{red}$ is a closed subscheme of $\mathfrak{B} + (Cl^{r+1})_E$.

5.1 CLUSTERS

We start fixing the notion of a family, the class of morphisms for which we will parametrise its families of clusters of sections.

Definition 5.1. Let S be a ground scheme and $\pi\colon X \longrightarrow Y$ a morphism. We call π an *S-family* if it is an S-morphism, fpqc and separated, the S-schemes X and Y are of finite type, Y is irreducible and the generic fibre of π is integral. The scheme Y is called the *base* and the scheme X the *ambient space*.

Example 5.2. When S is a base field $\Bbbk$ and X, Y are affine, then π corresponds to a faithfully flat homomorphism of $\Bbbk$-algebras of finite type $\varphi\colon A \longrightarrow B$, where the nilradical η of A is a prime ideal and $B/\varphi(\eta)$ is integral.

This notion, for us, is still too wild. We need to impose some regularity conditions on the families that we consider (see Sections 2.2 and 5.2), which leads us to the notion of a steady family, Definition 5.5 below.

Definition 5.3. Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-family. We call π *splitting weakly linear* (or that π splits weakly linear), if all its sections (which are closed embeddings, see Proposition 1.49) are weakly linear (see Definition 2.13).

1 This is an almost ad-hoc definition imposing that "when we require it", blow ups commute with arbitrary base changes.

Lemma 5.4. *Let S be a ground scheme. Let $\pi: X \longrightarrow Y$ be a splitting weakly linear S-family. Let σ be a section of π. Then, the morphism $X_\sigma \longrightarrow Y$ is an S-family.*

Proof. The morphism $X_\sigma \longrightarrow Y$ is separated, surjective and, by Remark 2.14.1, flat. Finally, the generic fibre η of $X_\sigma \longrightarrow Y$ is integral because, by Proposition 2.14, η is a blow up of the (integral) generic fibre of π. □

Definition 5.5. Let S be a ground scheme. Let $\pi: X \longrightarrow Y$ be an S-family. We call π *long splitting weakly linear* if, for every sequence of blow ups

$$X_r \longrightarrow X_{r-1} \longrightarrow \ \dots \ \longrightarrow X$$

whose centre $C_i \subseteq X_i$ is the image of a section $X_i \longrightarrow Y$, the S-families $X_i \longrightarrow Y$ split weakly linear. We call π *steady* if it is universally long splitting weakly linear, that is, for every morphism $T \longrightarrow S$, the T-family $\pi_T : X_T \longrightarrow Y_T$ is long splitting weakly linear.

Example 5.6. If a closed subscheme Z of a locally Noetherian scheme X is local complete intersection, that is the quasi-coherent $\mathcal{O}_X$-ideal $\mathcal{I}$ corresponding to Z can be locally generated by a regular sequence, then the symmetric and the Rees algebra of $\mathcal{I}$ are isomorphic, (the original result is due to Huneke, see [34, Theorem 3.1, p.269], or more recently [14, Exercise 17.14 (a), p.445]). In a Noetherian ring an ideal generated by a regular sequence can be generated by a d-sequence (see [34, Examples of d-sequences (1), p.1] and [14, Exercise 17.6, p.442]). So, examples of steady S-families $\pi: X \longrightarrow Y$ are:

- π is smooth: Smooth is stable by base change and a section of a smooth morphism is local complete intersection,
- the schemes X and Y are smooth over S: A smooth subscheme of a smooth scheme is local complete intersection and smooth is stable by base change.

Definition 5.7. Let $\pi: X \longrightarrow Y$ be a separated morphism. Given a sequence of blow ups

$$X_{r+1} \longrightarrow X_r \longrightarrow \dots \longrightarrow X_2 \longrightarrow X_1 = X$$

whose i-th centre $C_i \subseteq X_i$ is the image of a section t_i of $X_i \longrightarrow Y$, we call the sequence $(t_1, \dots, t_r)$ an *ordered cluster over* π (or for short a cluster over π). We call the integer r the *length* of the cluster.

Let S be a ground scheme and $\pi: X \longrightarrow Y$ a separated S-morphism. Notice that, since a section of π is an S-morphism for free, the notion of clusters over π is independent from considering π as a morphism or as an S-morphism.

Usually, clusters over π are interpreted as a sequence of arbitrarily near Y-points of the Y-scheme X. From this point of view, families of clusters over π are parametrised by Y-schemes T and the clusters in the family are parametrised by the Y-points of T. That leads to Definition 5.8 below.

Our new, and more general, point of view consists in interpreting a cluster over π as a sequence of arbitrarily near sections of the S-morphism π. Hence now, families of clusters over π are parametrised by S-schemes T and the clusters in the family are parametrised by the S-points of T. This leads to Definition 5.9 below.

Definition 5.8. Let $X \longrightarrow Y$ be a separated morphism. Let T be a Y-scheme. A T*-family of clusters of points over* π (or of point-clusters for short) is an ordered cluster over the projection $X_T \longrightarrow T$.

Definition 5.9. Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be a separated S-morphism. Let $T \longrightarrow S$ be a morphism. A T*-family of clusters of sections over* π (or of section-clusters for short) is an ordered cluster over the base change $\pi_T\colon X_T \longrightarrow Y_T$ of π by $T \longrightarrow S$.

Although not immediately obvious, there is a vast difference between families of point-clusters and of section-clusters.

For example, consider clusters of length one. The scheme parametrising families of point-clusters over a Y-scheme $X \longrightarrow Y$ always exists, indeed it is simply the scheme X itself. In contrast, the scheme parametrising families of section-clusters over an S-morphism $\pi\colon X \longrightarrow Y$ is, when it exists, the universal section family of π, which is typically infinite dimensional (see Section 1.5)

As another example, consider the absolute case when S is the spectrum of a field $\mathbb{k}$. On one side, we consider clusters over a $\mathbb{k}$-scheme $X \longrightarrow \mathbb{k}$. So, every ordered pair of distinct closed points (p, q) of X can be identified with a cluster over X, simply consider the sequence (p, q'), where q' is the preimage of q by the blow up of X along p.

On the other hand, we consider clusters over a $\mathbb{k}$-morphism $\pi\colon X \longrightarrow Y$. So now, given an ordered pair of sections (σ, τ) of π, we may consider the strict transform $\tilde{\tau}$ of τ by the blow up X_σ of X along σ, but $\tilde{\tau}$ is not necessarily a section of $X_\sigma \longrightarrow Y$. In fact, it is not hard to see that $\tilde{\tau}$ determines a section of $X_\sigma \longrightarrow Y$ if and only if $\sigma \times_X \tau$ (the intersection of σ and τ in X) is an effective Cartier divisor of σ. This phenomenon motivates the following definition.

Definition 5.10. Let $\pi\colon X \longrightarrow Y$ by a morphism. We say that a pair of sections σ, σ' of π is *admissible* (or σ is admissible with respect to σ') if $\sigma \times_Y \sigma'$ is an effective Cartier divisor of σ, or equivalently, of σ'.

When the scheme Y is smooth integral and of dimension one, there are no restrictions on the admissible pairs of sections. We will see examples where the existence of non-admissible pairs of sections has drastic consequences. Namely, the dimension of the parametrising scheme may decrease as we enlarge the length of the clusters to parametrise, see Example 5.40. As an immediate consequence, given integers $s < r$, in general we will *not* be able to recover the scheme parametrising section-clusters of length s from that of length r.

We finish defining some elementary but handy manipulations of clusters.

Definition 5.11. Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-scheme. Let $t = (t_1, \ldots, t_r)$ be a cluster over π and

$$X_{r+1} \longrightarrow X_r \xrightarrow{b} X_{r-1} \longrightarrow \ \ldots \ \longrightarrow X_2 \longrightarrow X_1 = X$$

its corresponding sequence of blow ups. The *switch down of* t, denoted by $t{\downarrow}$, is the length-$(r-1)$ cluster $(t_1, \ldots, t_{r-2}, b \circ t_r)$ over π.

Given a section $t_{r+1}\colon Y \longrightarrow X_{r+1}$ of $X_{r+1} \longrightarrow Y$, the *extension of* t *by* t_{r+1}, denoted by $t \sqcup t_{r+1}$, is the length-$(r+1)$ cluster $(t_1, \ldots, t_r, t_{r+1})$ over π.

Given an integer $1 \leqslant s \leqslant r$, the *truncation of* t *at* s, denoted by $t|_s$, is the length-s cluster $(t_1, \ldots, t_s)$ over π.

Notice that, given another cluster $t' = (t'_1, \ldots, t'_r)$ over π with sequence of blow ups

$$X'_{r+1} \longrightarrow X'_r \longrightarrow \ \ldots \ \longrightarrow X'_2 \longrightarrow X'_1 = X$$

if, for some $s \leqslant r$, the truncations $t|_s$ and $t'|_s$ are equal, then $X_i = X'_i$ for all $i = 1, \ldots, s+1$. That allows the following definition.

Definition 5.12. Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be an S-scheme. Let $t = (t_1, \ldots, t_r)$, $t' = (t'_1, \ldots, t'_r)$ be clusters over π with $t|_{r-1} = t'|_{r-1}$. We say that the pair t, t' is *admissible* (or t admissible with respect to t') if the pair of sections t_r, t'_r is admissible.

5.2 PARAMETRISING FAMILIES OF CLUSTERS

Let $\pi\colon X \longrightarrow Y$ be a separated morphism. Kleiman [38, Section 4.1, p.36] constructed inductively a sequence of (separated) maps $f_r\colon X_{r+1} \longrightarrow X_r$ for $r \geqslant 0$ as follows.[1] Define $f_0\colon X_1 \longrightarrow X_0$ to be $\pi\colon X \longrightarrow Y$. Now, assume f_{r-1} defined. Consider the Cartesian product of X_r with itself over X_{r-1} and consider its diagonal subscheme Δ, which is a closed subscheme because f_{r-1} is separated. Define X_{r+1} to be the corresponding residual scheme (see introduction to Chapter 2) and define f_r to be the composition of the structure map p and the second projection p_2.

$$\begin{tikzcd}
 & X_{r+1} = R(\Delta, X_r \times_{X_{r-1}} X_r) \arrow[d, "p"] & \\
X_r \arrow[d, "f_{r-1}"'] & X_r \times_{X_{r-1}} X_r \arrow[l, "p_1"'] \arrow[d, "p_2"] \arrow[dl, phantom, "\urcorner"] & \Delta \arrow[l, hook'] \\
X_{r-1} & X_r \arrow[l, "f_{r-1}"] &
\end{tikzcd}$$

In this section, we generalise Kleiman's construction of iterated blow ups to parametrise families of section-clusters. We define the functor for the parameter space of families of section-clusters, $\mathcal{Cl}$, and we show that it is

1 We reproduce the construction word for word as it is done in [38].

a subfunctor of a suitable Hilbert functor representable by locally closed embeddings, reducing its representability to that of such a Hilbert functor.

In general, families of section-clusters do not necessarily form a functor, nor even a category. Notice that this is already the case for families of point-clusters. In order to define the functor $\mathcal{Cl}$ or a morphism between families of section-clusters, we need to iteratively construct a sequence of morphisms (Proposition 5.14 synthesises both iterative steps). The obstruction we face is that blow ups do not commute with arbitrary base changes (see Section 2.2). Our procedure to overcome this difficulty is not original, indeed we will impose regularity conditions on the involved S-families (namely, we work over steady S-families, see Definition 5.5) as for example, Kleiman and Piene [41], who restrict to smooth morphisms or, more similar to us, Kleiman [38] who imposes conditions implying that blow ups are considered just along locally complete intersection closed subschemes. Nevertheless, our approach is slightly different. Steady S-families can be seen as an ad-hoc definition, imposing the weakest assumptions required to develop the whole theory (define the functor $\mathcal{Cl}$ and the category of families of section-clusters) from our approach. But, to our knowledge, the only examples of steady S-families are the ones given in the works of Kleiman and Piene, or very similar.

For convenience in the construction of a morphism between families of section-clusters and the functor for the parameter space of families of section-clusters, $\mathcal{Cl}$, we introduce the following two classes of morphisms.

Definition 5.13. Let S be a ground scheme. Let $\pi\colon X\longrightarrow Y$ be a separated S-morphism. Let $f\colon T'\longrightarrow T$ be an S-morphism. Let $(t_1,\dots,t_r)$ and $(t'_1,\dots,t'_r)$ be respectively T and T' families of cluster of sections over π. Given an integer $1\leqslant s\leqslant r$, we call a morphism $g\colon X'_s\longrightarrow X_s$ an s-*lift by sections of* f if the following diagram commutes.

$$\begin{tikzcd} Y_{T'} \arrow[r, "t'_s"] \arrow[d, "f_Y"'] & X'_s \arrow[d, "g"] \\ Y_T \arrow[r, "t_s"] & X_s \end{tikzcd} \tag{5.2.1}$$

Given an integer $1\leqslant s\leqslant r+1$, we call a morphism $g\colon X'_s\longrightarrow X_s$ an s-*lift by projections of* f (which is obviously unique, when it exists) if the following diagram is Cartesian.

$$\begin{tikzcd} X'_s \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y_{T'} \arrow[d, "f_Y"] \\ X_s \arrow[r] & Y_T \end{tikzcd} \tag{5.2.2}$$

Remark 5.13.1. If a morphism $g\colon X'_s\longrightarrow X_s$ is the s-lift by projections of f, then it is an s-lift by sections of f if and only if Diagram (5.2.1) is Cartesian, that is t'_s is the base change of t_s by f or, with another notation, $t'_s = \mathcal{S}ect_{X_s\longrightarrow Y_T} f(\sigma_s)$, where $X_s\longrightarrow Y_T$ is considered as a T-morphism (see Lemma 1.30).

Remark 5.13.2. The 1-lift by projections of $f\colon T' \to T$ always exists, it is f_X.

$$\begin{tikzcd} X_{T'} = X'_1 \arrow[r] \arrow[d, "f_X"'] \arrow[dr, phantom, "\ulcorner"] & Y_{T'} \arrow[d, "f_Y"] \\ X_T = X_1 \arrow[r] & Y_T \end{tikzcd}$$

Proposition 5.14. *Let* S *be a ground scheme and* $r \geqslant s \geqslant 1$ *integers. Let* $\pi\colon X \to Y$ *be a steady* S*-family. Let* $f\colon T' \to T$ *be an* S*-morphism. Let* $(t_1, \dots, t_r)$ *and* $(t'_1, \dots, t'_r)$ *be respectively* T *and* T' *families of section-clusters over* π*. If the* s*-lift by sections and projections* $g\colon X'_s \to X_s$ *of* f *exists, then the* $(s+1)$*-lift of* f *by projections also exists. We will denote it by* $\theta_f(g)\colon X'_{s+1} \to X_{s+1}$*.*

Proof. By Remark 5.13.1 and Proposition 2.14 the scheme X'_{s+1} is $X_{s+1} \times_T T'$ and then $\theta_f(g)$ is just the projection. □

Definition 5.15. Let S be a ground scheme. Let $\pi\colon X \to Y$ be a steady S-family. Let T and T' be S-schemes. Let $t = (t_1, \dots, t_r)$ and $t' = (t'_1, \dots, t'_r)$ be respectively T and T' families of cluster of sections over π. A *morphism of families of section-cluster* (or a cs-morphism for short), denoted by $f\colon (T', t') \to (T, t)$, is an S-morphism $f\colon T' \to T$ such that, for every $i = 1, \dots, r$, the morphism $f_i\colon X'_i \to X_i$, defined recursively as follows, is an i-lift by sections of f.

The initial morphism f_1 is $f_X\colon X_{T'} \to X_T$, which, by Remark 5.13.2, is the 1-lift by projections of f. The assumption that f_1 is the 1-lift by sections gives, by Proposition 5.14, the existence of the 2-lift by projections $\theta_f(f_1)\colon X'_2 \to X_2$ of f. Set $f_2 = \theta_f(f_1)$. Again, the assumption that f_2 is also the 2-lift by sections allows to iterate the process.

Clearly, cs-morphisms are stable by composition and identities are cs-morphisms. So, when π is steady, families of section-clusters over π form a category $\boldsymbol{Cl}_\pi$ where arrows are cs-morphisms.

Remark 5.15.1 below is the key point for the iterative construction of the schemes parametrising families of section-clusters. It shows that families of split sections are the step relating families of section-clusters of lengths r and $r+1$.

Remark 5.15.1. Let S be a ground scheme. Let $\pi\colon X \to Y$ be a steady S-family. Let T and T' be S-schemes and $t = (t_1, \dots, t_r)$ and $t' = (t'_1, \dots, t'_{r+1})$ be respectively T and T' families of cluster of sections over π. Given a cs-morphism $f\colon (T', t'|_r) \to (T, t)$, for all $i = 1, \dots, r$ the morphisms $f_i\colon X'_i \to X_i$ are the i-lift by sections and projections of f. In particular, so is $f_r\colon X'_r \to X_r$ and, by Proposition 5.14, the $(r+1)$-lift by projections $f_{r+1}\colon X_{r+1} \to X_{r+1}$ of f exists. The following diagram commutes,

$$\begin{tikzcd} Y_{T'} \arrow[r, "t_{r+1}"] \arrow[dr, "\mathbf{1}_{Y_{T'}}"'] & X'_r \arrow[r, "f_{r+1}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_r \arrow[d] \\ & Y_{T'} \arrow[r, "f_Y"] & Y_T \end{tikzcd}$$

and then the couple $(f_{r+1} \circ t_{r+1}, f)$ is a T'-family of T-split sections over $X_r \longrightarrow Y_T$. Moreover, the T'-family of T-split sections $(f_{r+1} \circ t_{r+1}, f)$ over π determines the section t_{r+1}, since $t_{r+1} = (f_{r+1} \circ t_{r+1}) \times_{Y_T} \mathbf{1}_{Y_{T'}}$.

Definition 5.16. Let S be a ground scheme and $r \geqslant 1$ an integer. Let $\pi\colon X \longrightarrow Y$ be a steady S-family. The *r-th universal scheme of families of section-clusters over π* (or r-Ucs for short), denoted by (Cl^r, τ^r), is the terminal object of the category of length-r families of section-clusters, $\boldsymbol{Cl}_\pi$. That is, Cl^r is an S-scheme, τ^r is a length-r Cl^r-family of section-clusters of over π and they satisfy the following universal property: For every S-scheme T and every length-r T-family of section-clusters t over π, there is a unique morphism $f\colon T \longrightarrow \mathrm{Cl}^r$ such that it is a cs-morphism $f\colon (T, t) \longrightarrow (\mathrm{Cl}^r, \tau^r)$.

If an r-Ucs over π exists, by abstract nonsense, it is uniquely determined up to a unique isomorphism. When the r-Ucs (Cl^r, τ^r) over π exists, we denote the corresponding sequence of blow ups by

$$X^r_{r+1} \xrightarrow{\mathrm{bl}^r_r} X^r_r \xrightarrow{\mathrm{bl}^r_{r-1}} \dots \xrightarrow{\mathrm{bl}^r_1} X^r_1 = X$$

and the composition $\pi \circ \mathrm{bl}^r_1 \circ \cdots \circ \mathrm{bl}^r_i : X^r_{i+1} \longrightarrow Y$ by π^r_{i+1}.

Definition 5.17. Let S be a ground scheme. Given a steady S-family $\pi\colon X \longrightarrow Y$, consider the contravariant functor $\mathcal{C}\ell^r_\pi\colon \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$ (sometimes we will omit the indices r or π) corresponding to the parameter space problem of families section-clusters over π of length r defined as follows. It sends an S-scheme T to the set of sequences of morphisms

$$\mathcal{C}\ell^r_\pi T = \{T\text{-families of section-clusters over } \pi \text{ of length } r\}.$$

Given an S-morphism $f\colon T' \longrightarrow T$, we build a map $\mathcal{C}\ell_\pi f$ sending a T-family of section-clusters $t = (t_1, \dots, t_r) \in \mathcal{C}\ell_\pi T$ to a T'-family of section-clusters $t' = (t'_1, \dots, t'_r) \in \mathcal{C}\ell_\pi T'$ for which $f\colon T' \longrightarrow T$ is a cs-morphism $f\colon (T', t') \longrightarrow (T, t)$. So, the category of elements of $\mathcal{C}\ell^r_\pi$ will be the category $\boldsymbol{Cl}_\pi$.

Consider the sequence of blow ups corresponding to t,

$$X_{r+1} \longrightarrow X_r \longrightarrow \dots \longrightarrow X_1 = X_T.$$

For $s = 1, \dots, r$, we construct recursively the section t'_s and the morphism f_s at the same time as follows. Assume that $t'|_s$ and the cs-morphism $f\colon (T', t'|_s) \longrightarrow (T, t|_s)$ are defined. Consider the sequence of blow ups corresponding to $t'|_s$,

$$X'_{s+1} \longrightarrow X'_s \longrightarrow \dots \longrightarrow X'_1 = X_{T'}.$$

In particular, by Remark 5.15.1, the $(s+1)$-lift by projections $f_{s+1}\colon X'_{s+1} \longrightarrow X_{s+1}$ of f is defined and we may consider the following Cartesian diagram.

$$
\begin{tikzcd}
& Y_{T'} \arrow[rr, "(t_{s+1})_{T'}"] \arrow[dl, "f_Y"'] \arrow[drr, "\mathbf{1}"] & & X'_{s+1} \arrow[dll, "f_{s+1}"'] \arrow[d] \\
Y_T \arrow[r, "t_{s+1}"] \arrow[dr, "\mathbf{1}"'] & X_{s+1} \arrow[d] & & Y_{T'} \arrow[dll, "f_Y"] \\
& Y_T & &
\end{tikzcd}
\tag{5.2.3}
$$

Setting $t'|_{s+1} = t'|_s \sqcup (t_{s+1})_{T'}$, that is $t'_{s+1} = (t_{s+1})_{T'}$, we have defined a length-$(s+1)$ T'-family of section-clusters and a cs-morphism

$$f\colon (T', t'|_{s+1}) \longrightarrow (T, t|_{s+1}).$$

Hence, we can iterate the process and, by Remark 5.13.2, we can initiate the construction with an empty T'-family of section-clusters and $f_1 = f_X$.

Remark 5.17.1. The functor $\mathcal{Cl}^r_\pi$ for $r = 1$ is equal to the functor of sections $\mathcal{Sect}_\pi$ of π. So, they are equivalently representable and, if they are representable, the representing schemes are isomorphic. That is, if they exist, the Usf $(\mathfrak{X}, \psi)$ of π and the 1-Ucs (Cl^1, τ^1) over π are isomorphic as families of sections over π, or equivalently, as families of section-clusters over π.

Remark 5.17.2. Consider π as a Y-morphism, where $Y \longrightarrow Y$ is the identity, let us denote π by $f\colon X \longrightarrow Y$ in order to distinguish this two cases. Assume that f is a steady Y-family. Given a Y-scheme T, a T-family of section-clusters over f is a T-family of point-clusters over f. So, the functor $\mathcal{Cl}^r_f\colon \boldsymbol{Sch}_Y \longrightarrow \boldsymbol{Set}$ is the functor for the parameter space problem of point-clusters over f. It is well known that, when they exist, Kleiman's iterated blow ups $b_r\colon X_r \longrightarrow Y$ represent the functor $\mathcal{Cl}^r_f$ (see [30, Proposition I.2, p.104] and [41, Proposition 3.4, p.422]). Setting by $g\colon Y \longrightarrow S$ the structure morphism of Y, the functor $\mathcal{Cl}^r_\pi$ is equal to $\mathcal{Cl}^r_f \circ \mathcal{P}_g$, where $\mathcal{P}_g$ is the base change functor (see Definition 1.33). Hence, by Proposition 1.68, $\mathcal{Cl}^r_\pi \cong \mathcal{Sect}_{b_r}$.

Theorem 5.18. *Let S be a ground scheme and $r \geqslant 1$ an integer. Let $\pi\colon X \longrightarrow Y$ be a steady S-family. Then, a family of section-clusters over π represents the functor $\mathcal{Cl}^r_\pi$ if and only if it is the r-Ucs of π.*

Proof. By construction, the category $\boldsymbol{Cl}_\pi$ of families of section-clusters over π with cs-morphisms as morphisms is the category of elements of $\mathcal{Cl}^r_\pi$. Hence, the claim follows from Proposition 1.15. ☐

Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be a steady S-family. Let T be an S-scheme and t a T-family of section-clusters over π. Then, to every S-point $s\colon S \longrightarrow T$ of T, we may associate a cluster t_s over π. Indeed, a cluster over π is just an S-family of section-clusters over π. Hence, since the composition $S \longrightarrow T \longrightarrow S$ is the identity, $\mathcal{Cl}^r(S \longrightarrow T)(t)$ is an actual cluster over π.

Corollary 5.18.1. *If the* r*-Ucs* (Cl^r, τ^r) *over* π *exists, the map sending an* S*-point* s *of* Cl^r *to the length-*r *cluster* τ^r_s *over* π *forms a one-to-one correspondence between* $\mathrm{Cl}^r(S)$ *and the set of length-*r *clusters over* π*.*

Theorem 5.19. *Let* S *be a locally Noetherian ground scheme and* $r \geqslant 1$ *an integer. Let* $\pi\colon X \longrightarrow Y$ *be both a steady* S*-family and a steady* Y*-family (considering* Y *as a* Y*-scheme via the identity). If* X *is at most a countable disjoint union of quasiprojective schemes over* S *and* Y *is proper and flat over* S*, then the* r*-Ucs* (Cl^r, τ^r) *over* π *exists and the scheme* Cl^r *is locally Noetherian and at most a countable disjoint union of quasiprojective schemes.*

Proof. Let us denote π as a Y-morphism by $f\colon X \longrightarrow Y$ in order to distinguish it. Consider Kleiman's iterated blow ups $b_r\colon X_r \longrightarrow Y$ for the morphism $X \longrightarrow Y$. In this situation the functor $\mathcal{Cl}^r_f\colon \boldsymbol{Sch}_Y \longrightarrow \boldsymbol{Set}$ is defined and the Y-scheme X_r represents it. Hence, by Remark 5.17.2, $\mathcal{Cl}^r_\pi \cong \mathcal{S}ect_{b_r}$. Observe that X_r is at most a countable disjoint union of quasiprojective scheme so, by Proposition 1.62, $\mathcal{S}ect_{b_r}$ is representable. □

Whereas Kleiman's iterated blow ups provide the representability of the relevant functors, the result is somewhat unsatisfactory, because it hides the relationship between Cl^{r+1} and Cl^r under the application of the functor of sections to Kleiman's iterated blow up. Now, we shall develop the machinery necessary for an iterative presentation of the schemes Cl^r, which will lead to a slightly more general second existence result.

Notation 5.20. Let S be a ground scheme. Let $\pi\colon X \longrightarrow Y$ be a steady S-family. Let T be an S-scheme and t a T-family of section-clusters over π of length $r+1$. Then, the truncation $t|_r$ and switch down $t{\downarrow}$ of t are T-families of section-clusters over π of length r. If the r-Ucs (Cl^r, τ^r) over π exists, truncation and switch down give rise to two cs-morphisms

$$p^r_t \;:\; (T, t|_r) \longrightarrow (\mathrm{Cl}^r, \tau^r)$$

$$b^r_t \;:\; (T, t{\downarrow}) \longrightarrow (\mathrm{Cl}^r, \tau^r).$$

For simplicity, in the particular case that the couple (T, t) is the $(r+1)$-Ucs of π, we will omit the subindex in the previous notation.

Remark 5.20.1. By the unicity of the cs-morphisms p^r_t and b^r_t, given a morphism $f\colon T' \longrightarrow T$, setting $t' = \mathcal{Cl}^{r+1}_\pi f(t)$,

$$p^r_{t'} = p^r_t \circ f \qquad\qquad b^r_{t'} = b^r_t \circ f.$$

Remark 5.20.2. If furthermore, the $(r-1)$-Ucs over π also exists, then $p^{r-1}_t \circ p^r_t = p^{r-1}_t \circ b^r_t$ as cs-morphisms.

Lemma 5.21. *Let* S *be a ground scheme and* $r \geqslant 1$ *an integer. Let* $\pi\colon X \longrightarrow Y$ *be a steady* S*-family. If the* r*-Ucs* (Cl^r, τ^r) *of* π *exists, then the functors* $\mathcal{Cl}^{r+1}_\pi$ *and* $\mathcal{S}ect^{\mathrm{Cl}^r}_{\pi^r_{r+1}}$ *(see Definition 3.24) are isomorphic. That is, given an* S*-scheme* T*, there is a one-to-one correspondence between* T*-families of* Cl^r*-split sections over* π^r_{r-1} *and* T*-families of section-clusters of length* $r+1$ *over* π*, and moreover it is natural on* T*.*

Proof. Fix an S-scheme T. We will construct a natural transformation

$$\eta\colon \mathcal{Cl}_\pi^{r+1} \longrightarrow \mathcal{S}ect_{\pi^r_{r-1}}^{\mathrm{Cl}^r}$$

and show it is a natural isomorphism by means of its inverse μ.

Given an element $t = (t_1, \dots, t_{r+1}) \in \mathcal{Cl}_\pi^{r+1}T$, by Remark 5.15.1, the couple

$$\Big(\big((p^r_t)_r \circ t_{r+1} \big),\ \ p^r_t \Big)$$

is an element of $\mathcal{S}ect_{\pi^r_r}^{\mathrm{Cl}^r}T$ (for $p^r_t\colon (T, t|_r) \longrightarrow (\mathrm{Cl}^r, \tau^r)$ see Notation 5.20). Set $\eta_T(t)$ as this couple. To check that this map is natural on T, fix an S-morphism $f\colon T' \longrightarrow T$ and set $t' = (t_1, \dots, t'_{r+1}) = \mathcal{Cl}^{r+1}f(t)$. By Remark 5.20.1, $p^r_{t'} = p^r_t \circ f$, and by functoriality of $\mathcal{Cl}^{r+1}$, $(p^r_{t'})_r = (p^r_t)_r \circ f_r$. Hence, $\eta_{T'}$ sends t' to the couple

$$\Big(\big((p^r_t)_r \circ f_r \circ t'_{r+1} \big),\ \ (p^r_t \circ f) \Big).$$

For the other side, by definition, the map $\mathcal{S}ect_{\pi^r_r}^{\mathrm{Cl}^r}f$ sends $\eta_T(t) = (((p^r_t)_r \circ t_{r+1}), p^r_t)$ to the couple $((p^r_t \circ t_r \circ f_Y), (p^r_t \circ f))$. So, we just need to check that $(p^r_t \circ f)_r \circ t'_r = p^r_t \circ t_r \circ f_Y$, which is clear by definition of t'_r, see Diagram (5.2.3).

An element of the set $\mathcal{S}ect_{\pi^r_r}^{\mathrm{Cl}^r}T$ is a T-family of Cl^r-split sections (σ, g) over $\pi^r_{r+1}\colon X^r_{r+1} \longrightarrow Y_T$, that is $\sigma\colon Y_T \longrightarrow X^r_{r+1}$ and $g\colon T \longrightarrow \mathrm{Cl}^r$ are S-morphisms such that

$$\pi^r_{r+1} \circ \sigma = g_Y.$$

Set $t = \mathcal{Cl}^r g(\tau^r)$ and $t_{r+1}\colon Y_T \longrightarrow X_{r+1}$ as the product of σ and $\mathbf{1}_{Y_T}$.

$$\begin{tikzcd}
Y_T \arrow[dr, dashed, "t_{r+1}"] \arrow[drr, bend left, "\mathbf{1}_Y"] \arrow[ddr, bend right, "\sigma"'] & & \\
& X_r \arrow[r] \arrow[d, "g_r"] \arrow[dr, phantom, "\ulcorner", very near start] & Y_T \arrow[d, "g_Y"] \\
& X^r_{r+1} \arrow[r, "\pi^r_{r+1}"] & Y_{\mathrm{Cl}^r}
\end{tikzcd}$$

By construction, t_{r+1} is a section of $X_r \longrightarrow Y_T$. Hence, the extension $t \sqcup t_{r+1}$ belongs to $\mathcal{Cl}_\pi^{r+1}(T)$. Set $\mu_T((\sigma, g)) = t \sqcup t_{r+1}$. To check that this map is natural on T, fix an S-morphism $f\colon T' \longrightarrow T$. By definition, the image of the couple (σ, g) by $\mathcal{S}ect_{\pi^r_r}^{\mathrm{Cl}^r}f$ is the couple $((\sigma \circ f_Y), g \circ f)$. On the other hand, setting $t' = \mathcal{Cl}^{r+1}f(t \sqcup t_r) = (t'_1, \dots, t'_{r+1})$,

$$\mathcal{Cl}^{r+1}f(t \sqcup t_r) = \mathcal{Cl}^r f(t) \sqcup t'_{r+1} = \mathcal{Cl}^r(g \circ f)(t) \sqcup t'_{r+1}.$$

Hence, we just need to check that t'_{r+1} is the product of $\sigma \circ f_Y$ with $\mathbf{1}_{Y_{T'}}$, but this is clear because the following diagram commutes.

$$\begin{tikzcd}
Y_{T'} \arrow[r, "t'_{r+1}"] \arrow[d, "f_Y"] \arrow[dr, phantom, "\ulcorner", very near start] & X'_{r+1} \arrow[r] \arrow[d, "f_{r+1}"] \arrow[dr, phantom, "\ulcorner", very near start] & Y_{T'} \arrow[d, "f_Y"'] \\
Y_T \arrow[r, "t_{r+1}"] \arrow[dr, "\sigma"'] & X_{r+1} \arrow[r] \arrow[d, "g_{r+1}"] \arrow[dr, phantom, "\ulcorner", very near start] & Y_T \arrow[d, "g_Y"] \\
& X^r_{r+1} \arrow[r, "\pi^r_{r+1}"] & Y_{\mathrm{Cl}^r}
\end{tikzcd}$$

Now, by Remark 5.15.1, it is clear that η and μ are mutually inverse. $\square$

Theorem 5.22. *Let S be a locally Noetherian ground scheme and $r \geqslant 1$ an integer. Let $\pi : X \longrightarrow Y$ be a steady S-family. If X is at most a countable disjoint union of quasiprojective schemes over S and Y is proper and flat over S, then the r-Ucs (Cl^r, τ^r) over π exists and the scheme Cl^r is locally Noetherian and at most a countable disjoint union of quasiprojective schemes.*

Proof. By induction on r. For $r = 1$, Remark 5.17.1 and Proposition 1.63 say that the 1-Ucs over π exists. It is its Usf, which it is locally Noetherian and at most a countable disjoint union of quasiprojective schemes.

Now, Lemma 5.21 and Theorem 3.26 provide the induction step. $\square$

5.3 ELEMENTARY CONSTRUCTIONS

This section is devoted to two results on families of section-clusters. The first construction is the generalisation for families of section-clusters of Proposition 1.64. The other two results of Section 1.5.1 do not generalise to families of section-clusters since, in general, taking the preimage of the image of a closed subscheme enlarges it.

The second result of this section, which is its motivation, relates the parameter spaces for clusters of different lengths, so it is particular for families of section-clusters.

Theorem 5.23. *Let S be a ground scheme. Let $\pi : X \longrightarrow Y$ be a steady S-family. Let $T \longrightarrow S$ be a morphism. Assume that the r-th universal scheme of section-clusters (Cl^r, τ^r) of π exists. Set $\tau^r_T = \mathcal{Cl}^r_\pi(\mathrm{Cl}^r_T \longrightarrow \mathrm{Cl}^r)(\tau^r)$. Then the r-th universal scheme of section-clusters of the (steady) T-family $\pi_T : X_T \longrightarrow Y_T$ is $(\mathrm{Cl}^r_T, \tau^r_T)$.*

Proof. Given a T-scheme $T' \longrightarrow T$, clearly the image of the T-scheme $T' \longrightarrow T$ by the functor

$$\mathcal{Cl}^r_{\pi_T} : \boldsymbol{Sch}_T \longrightarrow \boldsymbol{Set}$$

and the image of the S-scheme $T' \longrightarrow T \longrightarrow S$ by the functor

$$\mathcal{Cl}^r_\pi : \boldsymbol{Sch}_S \longrightarrow \boldsymbol{Set}$$

agree. Finally, by the universal property of pullbacks, there is an isomorphism

$$\boldsymbol{Sch}_S(T', \mathrm{Cl}^r) \cong \boldsymbol{Sch}_T(T', \mathrm{Cl}^r_T)$$

natural on T'. Hence, the scheme Cl^r_T represents the functor $\mathcal{Cl}^r_{\pi_T}$. $\square$

Theorem 5.24. *Let S be a ground scheme. Let $\pi\colon X \to Y$ be a steady S-family. Assume that $s < r$ and that the r-th and s-th universal schemes of section-clusters (Cl^r, τ^r) and (Cl^s, τ^s) of π exist. Let $t\colon S \longrightarrow \mathrm{Cl}^s$ be an S-point of Cl^s. Consider the length-s cluster $\mathrm{t} = (t_1, \dots, t_s) = \mathcal{Cl}^s t(\tau^s)$ and denote its corresponding sequence of blow ups by*

$$X_{s+1} \longrightarrow X_s \longrightarrow \dots \longrightarrow X_1 = X.$$

Denote the composition $X_{s+1} \to Y$ by π_t. Consider also the following Cartesian square,

$$\begin{tikzcd} \mathrm{Cl}^{r-s}_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d, "t"] \\ \mathrm{Cl}^r \arrow[r, "p"] & \mathrm{Cl}^s \end{tikzcd}$$

where p is the composition $p^s \circ \dots \circ p^{r-1}$ (see Notation 5.20). Set $\tau^r_t = (\sigma_1, \dots, \sigma_r) = \mathcal{Cl}^r_\pi(\mathrm{Cl}^{r-s}_t \longrightarrow \mathrm{Cl}^r)(\tau^r)$ and $\tau^{r-s}_t = (\sigma_{s+1}, \dots, \sigma_r)$. Then, the couple $(\mathrm{Cl}^{r-s}_t, \tau^{r-s}_t)$ is the $(r-s)$-th universal scheme of section-clusters of $\pi_t\colon X_{s+1} \to Y$.

Proof. Given an S-scheme T and a length-$(r-s)$ T-family of section-clusters over π_t,

$$(u_{s+1}, \dots, u_r),$$

it can be extended uniquely to a length-r T-family of section-clusters over π,

$$(\sigma_1, \dots, \sigma_s, u_{s+1}, \dots, u_r).$$

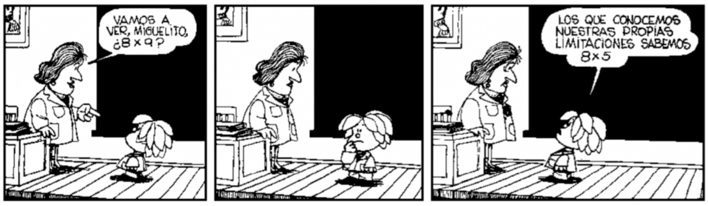
Hence, by the universal property of (Cl^r, τ^r), there is a unique morphism $T \to \mathrm{Cl}^r$ such that $\mathcal{Cl}^r_\pi(T \to \mathrm{Cl}^r)(\tau^r) = (\sigma_1, \dots, \sigma_s, u_{s+1}, \dots, u_r)$.

Observe that, since $\tau^r|_s = \mathcal{Cl}^s_\pi p(\tau^s)$, by functoriality of $\mathcal{Cl}^s_\pi$,

$$\mathcal{Cl}^s_\pi(\mathrm{Cl}^{r-s}_t \to S)(t) = (\sigma_1, \dots, \sigma_s).$$

So, by the universal property of (Cl^s, τ^s), the composition $T \longrightarrow \mathrm{Cl}^r \xrightarrow{p} \mathrm{Cl}^s$ is equal to $T \longrightarrow S \xrightarrow{t} \mathrm{Cl}^s$. Hence, there is a unique morphism $T \longrightarrow \mathrm{Cl}^{r-s}_t$ (the product of $T \to \mathrm{Cl}^r$ and $T \to S$) such that $\mathcal{Cl}^{r-s}_{\pi_t}(T \to \mathrm{Cl}^{r-s}_t)(\tau^{r-s}_t) = (u_{s+1}, \dots, u_r)$. $\square$

5.4 TOWARDS AN ITERATIVE CONSTRUCTION



– QUINO
Todo Mafalda

Fix a ground scheme S. Fix a steady S-family $\pi : X \longrightarrow Y$ with Y integral and $Y \longrightarrow S$ projective, smooth and with geometrically integral fibres. Assume that, for each $s = r-1, r, r+1$, the s-th universal scheme of section-clusters (Cl^s, τ^s) over π exists. So, we may consider the sequence of blow ups

$$X^s_{s+1} \xrightarrow{\mathrm{bl}^s_s} X^s_s \xrightarrow{\mathrm{bl}^s_{s-1}} \dots \longrightarrow X^s_1 = X_{\mathrm{Cl}^s},$$

whose j-th centre $C^s_j \subseteq X^s_j$ is the image of the section τ^s_j of $\pi^s_j : X^s_j \longrightarrow Y$. We will focus on the blow up $X^r_{r+1} \longrightarrow X^r_r$, so denote its exceptional divisor by E.

In this section, we show that the blow up §family is the iterative step to construct Cl^{r+1} from Cl^r. More precisely, there is a stratification of $\mathrm{Cl}_r \times_{\mathrm{Cl}_{r-1}} \mathrm{Cl}_r$ such that every irreducible component of Cl_{r+1} is either (a) birational to the closure of an irreducible component of a stratum or (b) composed entirely of clusters whose $(r+1)$-th section is infinitely near to the r-th, see Chapter 5 and Corollary 5.38.1. So, each type (a) irreducible component is an open subscheme of a blow up of an irreducible component of the closure of a stratum along a suitable sheaf of ideals. The blow up §family is the morphism from the union of all type (a) irreducible components (with a non-necessarily reduced structure) to the whole scheme $\mathrm{Cl}_r \times_{\mathrm{Cl}_{r-1}} \mathrm{Cl}_r$. That is, it incorporates the stratification of $\mathrm{Cl}_r \times_{\mathrm{Cl}_{r-1}} \mathrm{Cl}_r$ and strata-wise it is the corresponding blow up (see Theorem 5.37 and Corollary 5.38.1).

We also show that type (b) irreducible components form the Cl^r-Ussf of $E \longrightarrow Y_{\tau^r}$, so its general member corresponds to a cluster t over π whose $(r+1)$-th section t_{r+1} is a section of the corresponding exceptional divisor, which is not a flat limit of sections not contained in such exceptional divisor.

Notation 5.25. Consider Cl^r as an (Cl^{r-1})-scheme via the morphism p^{r-1}: $\mathrm{Cl}^r \longrightarrow \mathrm{Cl}^{r-1}$ (see Notation 5.20). We will use several times the scheme $\mathrm{Cl}^r \times_{\mathrm{Cl}^{r-1}} \mathrm{Cl}^r$, so we lighten up its notation to $(\mathrm{Cl}^r)^2$, and fix the notation $q_1, q_2 : (\mathrm{Cl}^r)^2 \longrightarrow \mathrm{Cl}^r$ for the projections over the first and the second factor respectively.

For $s = r-1, r$, consider an S-scheme T, a T-family of section-clusters $t = (t_1, \dots, t_{s+1})$ over π and its corresponding sequence of blow ups

$$X_{s+2} \longrightarrow X_{s+1} \longrightarrow X_s \longrightarrow \dots \longrightarrow X_1 = X_T.$$

The cs-morphism $p^s_t : (T, t|_s) \longrightarrow (\mathrm{Cl}^s, \tau^s)$ (see Notation 5.20) induces a morphism

$$(p^s_t)_{s+1} : X_{s+1} \longrightarrow X^s_{s+1}.$$

We denote by γ_t the composition

$$Y_T \xrightarrow{t_{s+1}} X_{s+1} \xrightarrow{(p^s_t)_{s+1}} X^s_{s+1}.$$

For $s = r$, we denote by E_t (which is a locally principal subscheme of Y_T) the pullback by $\gamma_t : Y_T \longrightarrow X^r_{r+1}$ of the exceptional divisor E in X^r_{r+1}.

In the particular case that (T, t) is the $(r+1)$-Usc $(\mathrm{Cl}^{r+1}, \tau^{r+1})$ of π, we denote $\gamma_{\tau^{r+1}}$ simply by $\gamma : Y_{\mathrm{Cl}^{r+1}} \longrightarrow X^r_{r+1}$ and $E_{\tau^{r+1}}$ by E_τ.

Observe that by Lemma 5.21 and Remark 3.27.1, the triplet $(\mathrm{Cl}^{r+1}, \gamma, p^r)$ is the Cl^r-Ussf of $X^r_{r+1} \longrightarrow Y_{\mathrm{Cl}^r}$.

Lemma 5.26. *The scheme* $(\mathrm{Cl}^r)^2$ *represents the functor* $\mathcal{S}ect^{\mathrm{Cl}^r}_{\pi^r_r}$.

Proof. Given an S-scheme T, we build a bijective map

$$\eta_T : \mathcal{S}ect^{\mathrm{Cl}^r}_{\pi^r_r}(T) \longrightarrow \boldsymbol{Sch}_S(T, (\mathrm{Cl}^r)^2),$$

which will be obviously natural on T.

An element of $\mathcal{S}ect^{\mathrm{Cl}^r}_{\pi^r_r}(T)$ is a couple of morphisms (β, f) with $\beta : Y_T \longrightarrow X^r_r$ and $f : T \longrightarrow \mathrm{Cl}^r$ such that $\pi^r_r \circ \beta = f_Y$. Observe that the couple $((p^{r-1})_r \circ \beta, (p^{r-1} \circ f))$ is a T-family of (Cl^{r-1})-split sections over π^{r-1}_r. Hence, since by Lemma 5.21 and Remark 3.27.1, the triplet $(\mathrm{Cl}^r, \gamma_{\tau^r}, p^{r-1})$ is the (Cl^{r-1})-Ussf of $\pi^{r-1}_r : X^{r-1}_r \longrightarrow Y_{\mathrm{Cl}^{r-1}}$, there is a unique morphism $g : T \longrightarrow \mathrm{Cl}^r$ such that

$$(p^{r-1})_r \circ \beta = \gamma_{\tau^r} \circ g_Y \tag{5.4.1}$$

(and $p^{r-1} \circ f = p^{r-1} \circ g$).

So, we set $\eta_T(\beta, f)$ as $(f \times_{\mathrm{Cl}^{r-1}} g) : T \longrightarrow (\mathrm{Cl}^r)^2$. Now, it is clear that f is determined by the morphism $T \longrightarrow (\mathrm{Cl}^r)^2$. Finally, by Remark 5.15.1 and Equation (5.4.1), the morphism $T \longrightarrow (\mathrm{Cl}^r)^2$ also determines β. □

Notation 5.27. Consider the natural transformation

$$\eta : \mathcal{S}ect^{\mathrm{Cl}^r}_{\pi^r_r} \longrightarrow \boldsymbol{Sch}_S(T, (\mathrm{Cl}^r)^2)$$

defined in the proof of Lemma 5.26. The preimage of $\mathbf{1}_{(\mathrm{Cl}^r)^2}$ by $\eta_{(\mathrm{Cl}^r)^2}$ is a couple of morphisms, $Y \times_S (\mathrm{Cl}^r)^2 \longrightarrow X^r_r$ and $(\mathrm{Cl}^r)^2 \longrightarrow \mathrm{Cl}^r$. By construction, the morphism $(\mathrm{Cl}^r)^2 \longrightarrow \mathrm{Cl}^r$ is the projection q_1 (recycling notation, if we define $\tilde{\eta}_T(\beta, f) = g \times_{\mathrm{Cl}^{r-1}} f$, then the morphism $(\mathrm{Cl}^r)^2 \longrightarrow \mathrm{Cl}^r$ in the preimage of $\mathbf{1}_{(\mathrm{Cl}^r)^2}$ by $\tilde{\eta}_{(\mathrm{Cl}^r)^2}$ is q_2). We denote the other morphism by $\rho : Y \times_S (\mathrm{Cl}^r)^2 \longrightarrow X^r_r$.

Note that the couple $(\mathrm{bl}^r_r \circ \gamma, p^r)$ is an (Cl^{r+1})-family of Cl^r-split sections over π^r_r, that is $\pi^r_r \circ (\mathrm{bl}^r_r \circ \gamma) = (p^r)_Y$. Hence, by Lemma 5.26, there is a unique morphism $F\colon \mathrm{Cl}^{r+1} \longrightarrow (\mathrm{Cl}^r)^2$ such that the following diagram commutes.

$$\begin{tikzcd} Y_{\mathrm{Cl}^{r+1}} \arrow[r, "\gamma"] \arrow[d, "F_Y"] & X^r_{r+1} \arrow[d, "\mathrm{bl}^r_r"] \\ Y \times_S (\mathrm{Cl}^r)^2 \arrow[r, "\rho"] & X^r_r \end{tikzcd}$$

The morphisms ρ and F could seem obscure, but they reveal its nature once considered acting over S-points. We may identify an S-point c of $(\mathrm{Cl}^r)^2$ with an ordered pair of clusters (t, t') over π such that $t|_{r-1} = t'|_{r-1}$. An S-point of X^r_r may be identified with a couple (u, p) where u is a length-r cluster over π and p is a point of the last but one scheme of the corresponding sequence of blow ups of u. So, slightly abusing notation, given an S-point y of Y,

$$\rho(y, c) = (t, t'_r(y)).$$

An S-point of Cl^{r+1} may be identified with a length-$(r+1)$ cluster ν over π, so

$$F(\nu) = (\nu|_r, \nu\!\downarrow).$$

Moreover, following with this notation,

$$\gamma(y, \nu) = (\nu|_r, \nu(y)).$$

We fix the following notation for every S-scheme T and T-family of section-clusters t over π.

Notation 5.28. We denote by T_B the closed subscheme of T for which the closed embedding $Y_{T_B} \hookrightarrow Y_T$ is the blow up of Y_T along E_t (see Theorem 2.8).

Notation 5.29. We denote by T_E the closed subscheme of T representing the functor $\mathit{Iso}^{E_t}_{Y_T \longrightarrow T}$ (see Theorem 3.17).

We denote by γ_B and γ_E the respective restrictions of $\gamma\colon Y_{\mathrm{Cl}^{r+1}} \longrightarrow X^r_{r+1}$ to $Y_{(\mathrm{Cl}^{r+1})_B}$ and $Y_{(\mathrm{Cl}^{r+1})_E}$.

Theorem 5.30. *Let $p_E\colon (\mathrm{Cl}^{r+1})_E \longrightarrow \mathrm{Cl}^r$ denote the restriction of $p^r\colon \mathrm{Cl}^{r+1} \longrightarrow \mathrm{Cl}^r$ to $(\mathrm{Cl}^{r+1})_E$. The triplet $((\mathrm{Cl}^{r+1})_E, \gamma_E, p_E)$ is the Cl^r-Ussf of $\pi_E\colon E \longrightarrow Y_{\mathrm{Cl}^r}$.*

Proof. We will check that it satisfies the required universal property. Given an S-scheme T and a T-family of Cl^r-split sections (σ, f) over $E \longrightarrow Y_{\mathrm{Cl}^r}$ (that is $\sigma\colon Y_T \longrightarrow E$, $f\colon T \longrightarrow \mathrm{Cl}^r$ such that $f_Y = \pi_E \circ \sigma$), composing σ with the closed embedding $E \hookrightarrow X^r_{r+1}$, it determines a unique T-family of Cl^r-split section family of $X^r_{r+1} \longrightarrow Y_{\mathrm{Cl}^r}$. Hence, by the universal property of the triplet $(\mathrm{Cl}^{r+1}, \gamma, p^r)$, there is a unique morphism $g\colon T \longrightarrow \mathrm{Cl}^{r+1}$ such that $\gamma \circ g_Y$ is the composition of σ with $E \hookrightarrow X^r_{r+1}$ (and $f = p^r \circ g$).

The base change of $E_\tau \hookrightarrow Y_{\mathrm{Cl}^{r+1}}$ by g is exactly the pullback of $E \hookrightarrow X^r_{r+1}$ by $\gamma \circ g_Y$. Hence, since $\gamma \circ g_Y$ factorises through $E \hookrightarrow X^r_{r+1}$ via σ, by

Lemma 1.32 such a base change is an isomorphism. Then, by the universal property of the closed embedding $(\mathrm{Cl}^{r+1})_B \hookrightarrow \mathrm{Cl}^{r+1}$ (inherited from representing the functor *Iso*), there is a unique morphism $T \longrightarrow (\mathrm{Cl}^{r+1})_B$ satisfying the required property. $\square$

Theorem 5.31. *The couple* $(\mathrm{Cl}_B^{r+1}, \gamma_B)$ *satisfies the following universal property. For every* S*-scheme* T *and* T*-family of section-clusters* t *over* π *such that* E_t *is an effective Cartier divisor of* Y_T*, there is a unique morphism* $f\colon T \longrightarrow \mathrm{Cl}_B^{r+1}$ *such that* $\gamma_t = \gamma_B \circ f_Y$*.*

Proof. Consider an S-scheme T and a T-family of section-clusters t over π such that E_t is an effective Cartier divisor.

By the universal property of $(\mathrm{Cl}^{r+1}, \tau^{r+1})$, there is a unique morphism $g\colon T \longrightarrow \mathrm{Cl}^{r+1}$ such that $t = \mathcal{C}\ell_\pi^r f(\tau^{r+1})$, in particular $\gamma_t = \gamma \circ g_Y$.

By the universal property of the blow up $Y_{(\mathrm{Cl}^{r+1})_B} \hookrightarrow Y_{\mathrm{Cl}^{r+1}}$, there is a unique morphism $h\colon Y_T \longrightarrow Y_{(\mathrm{Cl}^{r+1})_B}$ such that g_Y is the composition of h with $Y_{(\mathrm{Cl}^{r+1})_B} \hookrightarrow Y_{\mathrm{Cl}^{r+1}}$. Finally, Lemma 2.7 asserts that there is a unique morphism $f\colon T \longrightarrow (\mathrm{Cl}^{r+1})_B$ such that $h = f_Y$. $\square$

Proposition 5.32. *Let* T *be an* S*-scheme. Let* t *be a* T*-family of section-clusters over* π*. There are unique morphisms* $g_B\colon T_B \longrightarrow (\mathrm{Cl}^{r+1})_B$ *and* $g_E\colon T_E \longrightarrow (\mathrm{Cl}^{r+1})_E$ *such that* $\gamma_t|_{Y\times_S T_B} = \gamma_B \circ (g_B)_Y$ *and* $\gamma_t|_{Y\times_S T_E} = \gamma_E \circ (g_E)_Y$*.*

Proof. Setting $t_B = \mathcal{C}\ell_\pi(T_B \hookrightarrow T)(t)$, the morphism γ_{t_B} is the composition of γ_t with the closed embedding $T_B \hookrightarrow T$. By construction E_{t_B} is an effective Cartier divisor of Y_{T_B}, hence the existence and uniqueness of g_B comes from Theorem 5.31.

Since the pullback of $E_t \hookrightarrow Y_T$ by Y_{T_E} is an isomorphism, we may consider its inverse. It gives a T_E-family of Cl^r-split sections of $E \longrightarrow Y_{\mathrm{Cl}^r}$. Hence, the existence and uniqueness of g_E comes from Theorem 5.30. $\square$

Let us fix a bit of additional notation. Consider the blow up §family $(\mathfrak{B}, b)$ of the projection $Y \times_S (\mathrm{Cl}^r)^2 \longrightarrow (\mathrm{Cl}^r)^2$ along $\rho^{-1}(\mathrm{Im}(\tau_r^r))$. Notice that $\mathrm{Im}(\tau_r^r)$ is the centre of the blow up $X_{r+1}^r \longrightarrow X_r^r$. By construction, its preimage by $b_Y\colon Y_{\mathfrak{B}} \longrightarrow Y \times_S (\mathrm{Cl}^r)^2$ is an effective Cartier divisor. Hence, by the universal property of the blow up $X_{r+1}^r \longrightarrow X_r^r$, there is a unique morphism $\beta\colon Y_{\mathfrak{B}} \longrightarrow X_{r+1}^r$ such that $\beta = \rho \circ b_Y$.

Now, the couple $(\beta, q_1 \circ b)$ is a $\mathfrak{B}$-family of Cl^r-split sections of $X_{r+1}^r \longrightarrow Y_{\mathrm{Cl}^r}$. Hence, by the universal property of $(\mathrm{Cl}^{r+1}, \gamma, p^r)$ there is a unique morphism $G\colon \mathfrak{B} \longrightarrow \mathrm{Cl}^{r+1}$ such that $\beta = \gamma \circ G_Y$.

Lemma 5.33. *The morphisms* b, F *and* G *satisfy the relation* $b = F \circ G$*.*

Proof. Since $\rho \circ b_Y = \rho \circ F_Y \circ G_Y = \rho \circ (F \circ G)_Y$, by the universal property of $((\mathrm{Cl}^r)^2, \rho, q_1)$, $b = F \circ G$. $\square$

Proposition 5.34. *The core of the blow up §family of* $(\mathfrak{B}, b)$ *is the diagonal* Δ *of* $(\mathrm{Cl}^r)^2$*.*

Observe that, identifying an S-point c of $(\mathrm{Cl}^r)^2$ with a pair of clusters t, t' over π with $t|_{r-1} = t|_{r-1}$, the fibre of $\rho^{-1}(\mathrm{Im}(\tau^r_r)) \longrightarrow (\mathrm{Cl}^r)^2$ at c is isomorphic to $\mathrm{Im}(t_r) \cap \mathrm{Im}(t'_r)$. By Remark 3.14.2, the underlying set of such a core and Δ have to be equal.

Proof of Proposition 5.34. We show that the closed embedding $i\colon \Delta \hookrightarrow (\mathrm{Cl}^r)^2$ represents the functor $\mathit{Iso}^{\rho^{-1}(\mathrm{Im}(\tau^r))}_{Y\times(\mathrm{Cl}^r)^2 \longrightarrow (\mathrm{Cl}^r)^2}$. By construction, the image of $\rho \circ i_Y$ is equal to that of τ^r_r, so $\rho \circ i_Y$ factorises through $\mathrm{Im}(\tau^r_r)$ and then, by Lemma 1.32, the pullback of $\mathrm{Im}(\tau^r_r) \hookrightarrow X^r_r$ by $\rho \circ i_Y$ (which is the base change of $\rho^{-1}(\mathrm{Im}(\tau^r_r)) \hookrightarrow Y \times_S (\mathrm{Cl}^r)^2$ by i) is an isomorphism.

Now, given a morphism $f\colon T \longrightarrow (\mathrm{Cl}^r)^2$ such that $(\rho^{-1}(\mathrm{Im}(\tau^r_r)))_T \longrightarrow Y_T$ is an isomorphism, the composition of $q_1 \circ f\colon T \longrightarrow \mathrm{Cl}^r$ with $\Delta \hookrightarrow (\mathrm{Cl}^r)^2$ is equal to f. □

Theorem 5.35. *The couple $(\mathfrak{B}, \beta)$ satisfies the same universal property as the couple $((\mathrm{Cl}^{r+1})_B, \gamma_B)$ (see Theorem 5.31).*

Proof. Consider an S-scheme T and a T-family of section-clusters t over π such that E_t is an effective Cartier divisor.

By the universal property of $(\mathrm{Cl}^{r+1}, \tau^{r+1})$, there is a unique morphism $g\colon T \longrightarrow \mathrm{Cl}^{r+1}$ such that $t = \mathcal{Cl}^r_\pi f(\tau^{r+1})$, in particular $\gamma_t = \gamma \circ g_Y$.

The pullback of $\rho^{-1}(\mathrm{Im}(\tau)^r_r)$ by the composition $F_Y \circ g_Y$ is E_t, an effective Cartier divisor of Y_T by assumption (see Notation 5.27). Hence, by the universal property of the blow up §family $(\mathfrak{B}, b)$, there is a unique morphism $f\colon T \longrightarrow \mathfrak{B}$ such that $F \circ g = b \circ f$. Now, assuming $g_Y = G_Y \circ f_Y$, and then

$$\beta \circ f_Y = \gamma \circ G_Y \circ f_Y = \gamma \circ g_Y = \gamma_t.$$

To check that $g_Y = G_Y \circ f_Y$, by the universal property of $(\mathrm{Cl}^{r+1}, \gamma, p^r)$, we just need to check that their compositions with γ agree. So now, we are wondering whether the two morphisms, $\gamma \circ g_Y$ and $\gamma \circ G_Y \circ f_Y$ from Y_T to X^r_{r+1}, agree. But, by the universal property of the blow up $X^r_{r+1} \longrightarrow X^r_r$, we just need to check that their compositions with this blow up morphism agree, which is straightforward. □

Corollary 5.35.1. *The morphism $G\colon \mathfrak{B} \longrightarrow \mathrm{Cl}^{r+1}$ is a closed embedding.*

Proposition 5.36. *Let T be an S-scheme. Let t be a T-family of sections-clusters over π. If T is integral, then it is equal to either T_B or T_E.*

Proof. Since, by assumption, T and Y are integral, the locally principal subscheme E_t of Y_T is either an effective Cartier divisor or the whole scheme Y_T. So, in the former case the blow up $Y_{T_B} \hookrightarrow Y_T$ is an isomorphism and in the latter the scheme T itself represents the functor $\mathit{Iso}^{E_t}_{Y_T \longrightarrow T}$. □

Theorem 5.37. *Let T be an S-scheme. Let t be a T-family of sections-clusters over π. The scheme T_{red} is a closed subscheme of the schematic union $T_E + T_B$. In particular, the underlying topological spaces of $\mathfrak{B} + (\mathrm{Cl}^{r+1})_E$ and Cl^{r+1} are homeomorphic.*

Proof. By Proposition 5.36 every irreducible component of T, with its reduced structure, is a closed subscheme of either T_B or T_E. □

Consider the flattening stratification

$$(\mathrm{Cl}^r)^2 = \bigsqcup_{\Phi} (\mathrm{Cl}^r)^2_{\Phi}$$

of $\rho^{-1}(\mathrm{Im}(\tau^r_r)) \longrightarrow (\mathrm{Cl}^r)^2$ (see Section 1.6). From now on, assume Y smooth over S. So, the connected components of the strata are type I or type II, see Definition 4.6.

By Proposition 5.34, $\Delta \hookrightarrow (\mathrm{Cl}^r)^2$ is a type II stratum. Slightly abusing notation let us set

$$(\mathrm{Cl}^r)^2 = \Delta \sqcup \bigsqcup_{\Phi \in \Omega'} (\mathrm{Cl}^r)^2_{\Phi} \tag{5.4.2}$$

as the disjoint union of the connected components of the strata and call its components strata again. By Theorem 4.8 and Corollary 4.8.2, denoting by Ω the set of indices for all type I strata, b gives an isomorphism

$$\mathfrak{B} \setminus b^{-1}(\Delta) \cong \bigsqcup_{\Phi \in \Omega} (\mathrm{Cl}^r)^2_{\Phi}.$$

Proposition 5.38. *An S-point c of a type I stratum $(\mathrm{Cl}^r)^2_{\Phi}$ corresponds to an admissible pair of clusters over π (see Definition 5.12).*

Proof. Since $(\mathrm{Cl}^r)^2_{\Phi}$ is separated over S, c is a closed embedding. So, clearly by construction and by Lemma 4.4 (a), the base change $\rho^{-1}(\mathrm{Im}(\tau^r_r))_c \hookrightarrow Y$ of $\rho^{-1}(\mathrm{Im}(\tau^r_r)) \longrightarrow Y \times_S (\mathrm{Cl}^r)^2_{\Phi}$ by $c\colon S \longrightarrow (\mathrm{Cl}^r)^2_{\Phi}$ is an effective Cartier divisor. □

Corollary 5.38.1. *Each irreducible component Z of Cl^{r+1} is either*

1. *composed entirely of clusters whose $(r+1)$-th section is infinitely near to the r-th (that is, $Z \subseteq (\mathrm{Cl}^{r+1})_E$ and $F(Z) \subset \Delta$),*
2. *birational to an irreducible component of the closure C of a type I stratum, that is, $F|_Z\colon Z \longrightarrow C$ is a blowup map whose centre fails to be Cartier only on Δ. In particular, if $C \cap \Delta$ is empty, $Z \cong_F C$.*

Proof. Follows immediately from Corollary 4.8.2 and Theorem 5.37. □

We finish showing, with a small example, that in fact we expect the schemes $\mathfrak{B} + (\mathrm{Cl}^{r+1})_E$ and Cl^{r+1} to be isomorphic. Consider an S-scheme T and a T-family of section-clusters t over π. The scheme T_E is a closed subscheme of T, for which $\gamma_t|_{Y_{T_E}}$ is a T_E-family of Cl^r-split sections over $E \longrightarrow Y_{\mathrm{Cl}^r}$. Observe that, once we get the closed subscheme T_B of T, there is another natural (and maybe more intuitive) way to obtain a closed subscheme of T parametrising Cl^r-split sections of $E \longrightarrow Y_{\mathrm{Cl}^r}$. Namely, as the schematic closure of the open embedding $(T \setminus T_B) \hookrightarrow T$, let us call it T^{ii}_E. These two constructions are equivalent in some cases, but to our knowledge, in general there are just closed embeddings $T_{red} \hookrightarrow (T_B \cup T_E^{ii}) \hookrightarrow (T_B \cup T_E) \hookrightarrow T$. Let us show it with a couple of examples unrelated to section families.

We consider $Y \longrightarrow S$ the identity of the spectrum of a base field $\Bbbk$, so the projection $Y_T \longrightarrow T$ is just the identity and the scheme T_E is equal to E_t. One obtains an example where $T_E = T_E^{ii}$ by taking T equal to the spectrum of $A = \Bbbk[x,y]/(xy)$ and E_t is the principal subscheme determined by $x \in \Bbbk[x,y]/(xy)$. In this case, the closed embedding $T_B \hookrightarrow T$ corresponds to the natural homomorphism $A \longrightarrow A/(y)$ and both, T_E and T_E^{ii}, coincide the spectrum of $A/(x)$. But if we replace A with $\tilde{A} = \Bbbk[x,y]/(y^2, xy)$, then $\tilde{T}_E^{ii}$ is empty whereas the schemes $\tilde{T}_E + \tilde{T}_B$ and $\tilde{T}$ are equal.

5.5 EXAMPLES

In this section, we collect a few simple examples of r-Ucs for $r = 1, 2$ over families of surfaces (defined over a base field $\Bbbk$), whose behaviour differs from Kleiman's iterated blow ups in distinct ways. We consider families $\pi\colon X \longrightarrow Y$ with Y and X projective; by Theorem 5.19 the r-Ucs (Cl^r, τ^r) of π exists for all $r \geqslant 1$. Throughout this section, notation of Section 5.4 for $r = 1$ is fixed. Finally, if $(\mathfrak{X}, \psi)$ is the Usf of π, by Remark 5.17.1 the scheme Cl^1 is isomorphic to $\mathfrak{X}$. So, we will refer to the elements of Cl^1 as sections and of Cl^2 as clusters.

Example 5.39 (New components)**.** We show a family for which $(\mathrm{Cl}^2)_E$ has infinitely many connected components, whose clusters can *not* be obtained as the strict transform of flat limits of pairs of sections of π, or in other words, those are also connected components in Cl^2.

Consider a smooth projective family $\pi\colon X \longrightarrow Y$ of relative dimension 2 with Y a smooth curve.

The irreducible components of Cl^1 are classified by the degree of the images of sections of π and there are at most finitely many components for each degree. So, there are at most countably many irreducible components T_d of Cl^1 with $d \in \mathbb{N}$.

For any pair of integers $d, d' \geqslant 0$, there is a positive integer which bounds the degree of the 0-cycle intersection of any pair of sections in $T_d \times T_{d'}$. Given an integer $i \geqslant 0$, we denote by D_i the locally closed subscheme of $(\mathrm{Cl}^1)^2$ consisting of pairs of sections whose intersection is a 0-cycle of degree i. For each $i, d, d' \geqslant 0$, set $D_{i,d,d'} = D_i \cap (T_d \times T_{d'})$, which is either empty or an irreducible and connected component of D_i. So, the stratification Equation (5.4.2) is given by

$$(\mathrm{Cl}^1)^2 = \Delta \sqcup \bigsqcup_{i,d,d'} D_{i,d,d'}.$$

There are no restrictions on the admissible pairs of sections because the base is smooth and of dimension one. So, $\Delta \hookrightarrow (\mathrm{Cl}^1)^2$ is the unique type II stratum and for every $i, d, d' \geqslant 0$ there is an irreducible component $Z_{i,d,d'}$ of Cl^2 birational to the closure of $D_{i,d,d'}$.

Now assume, for simplicity, $Y = \mathbb{P}^1_{\mathbb{k}}$ for some field $\mathbb{k}$. For each section σ of π, the exceptional divisor E_σ in the blow up $X_\sigma \longrightarrow X$ is a rational surface isomorphic to the projectivisation of the normal bundle of σ in X, which (for some $a, b \in \mathbb{Z}$, say with $b \geqslant a$) is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$, the Hirzebruch surface F_{b-a}. So, setting $e = b - a$, there are two divisors C, F in E_σ such that

$$\operatorname{Pic}(E_\sigma) = \mathbb{Z}[C] + \mathbb{Z}[F]$$

with $C^2 = -e$, $F^2 = 0$ and $CF = 1$ (see [32, Chapter V, Theorem 2.17, p.379]). Observe that any irreducible curve of E_σ, say linearly equivalent to $D = nC + mF$ for some $n, m \in \mathbb{Z}$, intersects every fibre $(E_\sigma)_p$ at exactly one point for every $p \in \mathbb{P}^1_{\mathbb{k}}$ if and only if $1 = D \cdot F = n$. That is, an irreducible curve of E_σ is the image of a section of $X_\sigma \longrightarrow \mathbb{P}^1_{\mathbb{k}}$ if and only if it is irreducible and linearly equivalent to $C + mF$, with $m = 0$ or $m \geqslant e$. The sections of $E_\sigma \longrightarrow \mathbb{P}^1_{\mathbb{k}}$ are in correspondence with non-reduced schemes supported at σ in X, via direct images and strict transforms. Since σ has a fixed degree, there are finitely many possible degrees for non-reduced schemes supported at σ obtained as flat limits of pairs of sections of $X \longrightarrow \mathbb{P}^1_{\mathbb{k}}$.

Hence, there are infinitely many irreducible components in $(\mathrm{Cl}^2)_E$, one for each $m \geqslant e$ or $m = 0$, and infinitely many of them are filled up with clusters which are not limits of points of any $Z_{i,d,d'}$

Example 5.40 (The dimension may decrease). This example illustrates that the dimension of the schemes parametrising clusters may decrease as we enlarge the length of the clusters to parametrise. The phenomenon is due to the admissibility restriction on pairs of sections, which does not exist in the absolute setting, or when the base is a integral smooth curve.

Consider as a family the projection on the second factor $\pi \colon \mathbb{P}^2_{\mathbb{k}} \times \mathbb{P}^2_{\mathbb{k}} \longrightarrow \mathbb{P}^2_{\mathbb{k}}$, for some field $\mathbb{k}$. The scheme Cl^1 is a union of irreducible connected components C_d with $d \geqslant 0$, each one isomorphic to the open subscheme of $\mathbb{P}(\mathbb{k}[u, v, w]^3_d)$ corresponding to triplets of forms with no common roots.

Lemma 5.41. *Given two morphism $f, g \colon \mathbb{P}^2_{\mathbb{k}} \longrightarrow \mathbb{P}^2_{\mathbb{k}}$, with f non-constant, the intersection Z of the graphs of f and g in $\mathbb{P}^2_{\mathbb{k}} \times \mathbb{P}^2_{\mathbb{k}}$ is* not *an effective Cartier divisor of the graph of f.*

Proof. The graphs of f and g are varieties of dimension two in a four dimensional ambient space, hence in general they intersect in a codimension 2 subvariety. It is not hard to see that this is always the case. □

By Lemma 5.41, a pair of sections of π is admissible if and only if both sections are constant. Given a closed point c of Cl^2 with image $F(c) = (\sigma, \tau) \in (\mathrm{Cl}^1)^2$, the couple (σ, τ) can be either an admissible pair of sections of π (then, both σ, τ are constant) or τ is a section of the exceptional divisor E_σ in the blow up $X_\sigma \longrightarrow X$. If σ is constant, say with image $q \in \mathbb{P}^2_{\mathbb{k}}$, then $X_\sigma = \mathrm{bl}(q, \mathbb{P}^2) \times \mathbb{P}^2$ and $E_\sigma \longrightarrow \mathbb{P}^2_{\mathbb{k}}$ is isomorphic to the projection $\mathbb{P}^1_{\mathbb{k}} \times$

$\mathbb{P}^2_{\Bbbk} \longrightarrow \mathbb{P}^2_{\Bbbk}$, which admits only constant sections. Finally, when σ is non-constant, $E_\sigma \longrightarrow \mathbb{P}^2_{\Bbbk}$ admits no sections. Hence,

$$\mathrm{Cl}^2 = \mathrm{bl}\big(\Delta_{\mathbb{P}^2}, \mathbb{P}^2 \times \mathbb{P}^2\big).$$

Example 5.42 (Non-unique centre)**.** This example (a particular case of Example 5.39 explicitly computed) illustrates that the ideal sheaf, centre of a blow up $F|_Z \colon Z \longrightarrow C$ (see notations Corollary 5.38.1), need not be the ideal sheaf of $\Delta \cap C$ in C nor unique (in this case, the singularities of C allow different centres for the same blow up morphism).

Fix a line L in the three dimensional projective space $\mathbb{P}^3_{\Bbbk}$ over a field $\Bbbk$. We consider as a family the $\mathbb{P}^1_{\Bbbk}$ pencil of planes containing L. The ambient space is the quasiprojective variety $X = \mathbb{P}^3_{\Bbbk} \setminus L$. There are several equivalent ways to describe the projection $\pi \colon X \longrightarrow \mathbb{P}^1_{\Bbbk}$, via Grasmannians, the tangent space of a nondegenerated quadric containing L, the blow up $\mathrm{bl} : \mathrm{bl}(L, \mathbb{P}^3_{\Bbbk}) \longrightarrow \mathbb{P}^3_{\Bbbk}$ and more. Consider $\mathbb{P}^3_{\Bbbk}$ as the homogeneous spectrum of $R = \Bbbk[x, y, z, w]$ and, via a linear change of coordinates if it is needed, assume L cut out by $\mathfrak{a} = (x, y)$. So,

$$\begin{array}{rcl} \pi \ : \ X & \longrightarrow & \mathbb{P}^1_{\Bbbk} \\ {[x : y : z : w]} & \longmapsto & [x : y] \end{array}$$

and the fibre X_p for a point $p = [\alpha : \beta] \in \mathbb{P}^1_{\Bbbk}$ is the plane cut out by $\beta x - \alpha y$ minus L.

The image of a section of π is a projective rational curve, disjoint to L, that intersects every plane $\overline{X_p}$ at one point. So, it is simply a line disjoint to L and every line disjoint to L determines a section of π. Hence, Cl^1 is the open subvariety of the Grasmannian $\mathbb{G}$ of lines in $\mathbb{P}^3_{\Bbbk}$ corresponding to the lines not meeting L. This open subvariety is the complement of the tangent space of $\mathbb{G}$ at L, which is isomorphic to $\mathbb{A}^4_{\Bbbk}$, say with ring of functions $\Bbbk[a, b, c, d]$. We may pick a parametrisation which associates a point $\sigma = (a, b, c, d) \in \mathrm{Cl}^1$ the line

$$L_\sigma = \mathrm{Spec}\begin{pmatrix} ax + by - z \\ cx + dy - t \end{pmatrix} \subseteq X.$$

So, the morphism $\rho \colon \mathbb{P}^1_{\Bbbk} \times (\mathrm{Cl}^1)^2 \longrightarrow \mathrm{Cl}^1 \times X$ sends

$$\big([u : v], (a, b, c, d), (a', b', c', d')\big)$$

to

$$\big((a, b, c, d), [u : v : a'u + b'v : c'u + d'v]\big).$$

Set $W = V((a - a')(d - d') - (b - b')(c - c')) \subset \mathrm{Cl}^1 \times \mathrm{Cl}^1$. Given $p \neq p' \in \mathrm{Cl}^1$, the lines L_p, $L_{p'}$ meet if and only if $(p, p') \in W$ and in this case they always meet at a simple point. Hence, the flattening stratification Equation (5.4.2) is given by

$$\mathrm{Cl}^1 \times \mathrm{Cl}^1 = \Delta \sqcup (W \setminus \Delta) \sqcup W^c.$$

Now, we focus on the irreducible component Z of Cl^2 dominating the stratum $W \setminus \Delta$. The variety X_2^1 is given by

$$V\big(\mu(ax+by-z)-\nu(cx+dy-w)\big) \subseteq \mathrm{Cl}^1 \times X \times \mathbb{P}^1_{\Bbbk},$$

where $[\nu : \mu]$ are the coordinates of $\mathbb{P}^1_{\Bbbk}$. The morphism ρ restricted to the stratum $\mathbb{P}^1_{\Bbbk} \times (W \setminus \Delta)$ extends to X_2^1. Over the coordinates $[\nu : \mu]$, it is $[a-a' : b-b']$ or $[c-c' : d-d']$ depending on which is well-defined, and, in case both are, they are equal since $(a, b, c, d, a', b', c', d')$ belongs to W. But the morphism ρ does *not* extend to the diagonal. To see this, consider the blow up of $(\mathrm{Cl}^1)^2$ along the ideal $(a-a', b-b')$, that is

$$V(\eta(a-a')-\omega(b-b')) \subseteq (\mathrm{Cl}^1)^2 \times \mathbb{P}^1_{\Bbbk},$$

where $[\omega : \eta]$ are the coordinates of $\mathbb{P}^1_{\Bbbk}$. The strict transform $\tilde{W}$ of W under this blow up is a small resolution of W. Now, we can lift the morphism ρ to $\tilde{W} \times \mathbb{P}^1_{\Bbbk} \longrightarrow X_1^0$, over the coordinates $[\nu : \mu]$ it is just $[\omega : \eta]$. That implies $\tilde{W} \cong Z$, because any two distinct points of $\tilde{W}$ give two distinct sections of $X_2^1 \longrightarrow \mathbb{P}^1_{\Bbbk}$ and the dimensions agree. Observe that the ideal is not unique, the ideal $(c-c', d-d')$ also works.

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> So Matilda's strong young mind continued to grow, nurtured by the voices of all those authors who had sent their books out into the world like ships on the sea. These books gave Matilda a hopeful and comforting message: You are not alone.
>
> –Roald Dahl
> *Matilda*

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There is no conflict between the individual and the social instincts, any more than there is between the heart and the lungs: the one the receptacle of a precious life essence, the other the repository of the elements that keeps the essence pure and string. The individual is the heart of society, conserving the essence of social life; society is the lungs which are distributing the element to keep the life essence –that is, the individual– pure and strong.

–Emma Goldman
Anarchism: What It Really Stands For

My pussy tastes like Pepsi cola.
–Lana Del Rey
Cola

COLOPHON

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